Stability of differential operators on corrected surfaces
Master 2 Internship proposal

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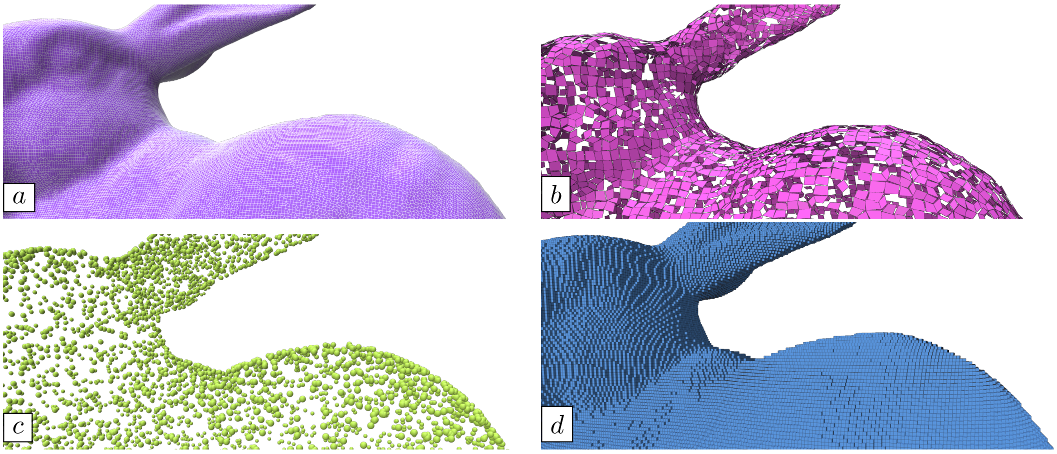

Figure 1: Discrete models approximating smooth surfaces embedded in $\mathbb{R}^3$: polygonal meshes with piecewise linear elements, without (a) or with topological defects (b); point clouds sampling the surface (c), or digital surface (d), i.e., the boundary of voxels within the object in $\mathbb{Z}^d$.

In digital geometry processing, whether it is for surface reconstruction, shape alignment, compression, denoising, parameterization, or more generally, the processing of scalar or vector fields defined on a 3D surface, we often have to solve variational problems on discrete structures (point clouds, embedded graphs, piecewise linear approximations - whether triangular or not, voxel grids, etc.), frequently affected by topological perturbations or inconsistencies (see Figure 1).

For this purpose, we need robust operators on these structures, in particular the counterparts of classical differential geometry operators (gradient, divergence, curl, Laplace-Beltrami, connections, etc.). Furthermore, we must also ensure certain properties on these operators through a notion of convergence to their continuous counterparts on smooth differential manifolds while preserving their algebraic properties.

The literature offers a rich toolbox of operators and schemes for such calculus on triangulated or quadrangulated meshes (discrete exterior calculus, finite elements, finite volumes, etc.). However, these operators exhibit a behavior similar to classical differential operators only on triangular or quadrangular surface data, with strong assumptions regarding vertex positions relative to the smooth surface or the shape of elements (reasonably regular triangles, close normal vectors).

The goal of this master's topic is to investigate a proposal for **differential calculus on discrete surfaces referred to as corrected**. A *corrected surface* is a discrete surface (triangular or polygonal mesh, digital surface) equipped with a corrected normal field, which may not necessarily be the normal vector field induced by vertex positions. This corrected field can, for example, provide a better estimation of normal vectors when the surface is noisy or irregularly sampled (e.g., the Schwarz lantern) or offer a convergent estimator of normal vectors in the case of digital surfaces (where trivial normals have only 6 possible directions). If we can prove the **stability** of this calculus, considering the error in the corrected position/normal fields concerning the underlying smooth surface, we then have a differential calculus that remains consistent across various discrete data. Thus, we can hope to solve differential problems on the different data types shown in Figure 1 with a common, similar computation framework, yielding results that are close in each case.

We will primarily study the **stability of the following discrete calculi**:

- The corrected calculus presented in [1], which is an adaptation of the polygonal calculus proposed in [2]. This calculus illustrates that it is possible to perform differential operations on digital surfaces with numerical results close to what one would obtain on a triangulated surface sampling the underlying smooth surface. This type of calculus is conceptually similar to discrete calculus [3], discrete exterior calculus [4], discrete differential calculus [5, 6], or the method of virtual elements, which aim to respect the structural properties of calculus (e.g., Stokes' theorem) rather than focusing on precision.

- The two corrected calculus presented in [7], one is a variant of the finite element method with correction through the metric, while the other formalizes the corrected calculus in the Grassmannian by interpolating position and normal fields. Experiments show that these variants of calculi yield results comparable to calculus on typical meshed surfaces. For example, there is a convergence of the Laplace-Beltrami operator similar to [8], whereas it was previously the only operator shown to strongly converge on digital surfaces.

The interpolated calculus in the Grassmannian seems to be a promising candidate for a generic corrected calculus because it defines its operators within the Grassmannian, a space that decouples positions and normals, allowing the embedding and comparison of various discrete geometric models. With this same approach, we were able to extend the normal cycle[9] to more general surfaces [10, 11], leading to the convergence of curvature measures (including point-to-point curvatures) on challenging surfaces such as digital surfaces or the Schwarz lantern. The numerical results were even better than the state of the art on typical meshed surfaces. Recently, we demonstrated that this framework also allows for the stable definition of curvatures on point clouds [12]. If this approach has proven effective for computing geometric differential quantities, we can hope that it will yield good results for differential calculus as well.

References

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