A Theoretical Analysis of Compactness of the Light Transport Operator

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GDL 15 Octobre 2025

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Rendering photorealistic visuals of virtual scenes requires tractable models for the simulation of light. The rendering equation describes one such model using an integral equation, the crux of which is a continuous integral operator. A majority of rendering algorithms aim to approximate the effect of this *light transport operator* via discretization (using rays, particles, patches, etc.). Research spanning four decades has uncovered interesting properties and intuition surrounding this operator. In this paper we analyze *compactness*, a key property that is independent of its discretization and which characterizes the ability to approximate the operator uniformly by a sequence of finite rank operators. We conclusively prove lingering suspicions that this operator is not compact and therefore that any discretization that relies on a finite-rank or nonadaptive finite-bases is susceptible to unbounded error over arbitrary light distributions. Our result justifies the expectation for rendering algorithms to be evaluated using a variety of scenes and illumination conditions. We also discover that its lower dimensional counterpart (over purely diffuse scenes) is not compact except in special cases, and uncover connections with it being noninvertible and acting as a low-pass filter. We explain the relevance of our results in the context of previous work. We believe that our theoretical results will inform future rendering algorithms regarding practical choices.

CCS Concepts: • **Computing methodologies → Rendering**; • **Mathematics of computing** → *Functional analysis; Integral equations.*

Additional Key Words and Phrases: Light Transport Operator, Compactness, Fredholm Equations

ACM Reference Format:
Cyril Soler, Ronak Molazem, and Kartic Subr. 2022. A Theoretical Analysis of Compactness of the Light Transport Operator. In *ACM Transations on Graphics*. ACM, New York, NY, USA, Article 31, 9 pages. https://doi.org/10.1145/3528233.3530725

1 INTRODUCTION

The simulation of light transport has a long history in the computer graphics literature. Several numerical approximations have been devised such as the use of rays [Ward et al. 1988], stochasticity [Cook 1986], finite patches [Goral et al. 1984; Hanrahan et al. 1991], particles [Jensen 1996] and paths of light [Veach 1997]. Elegant theories have confirmed that these methods all serve as approximations to a central equation that governs radiative transfer [Heckbert and Winget 1991; Kajiya 1986; Lessig 2012]. This *rendering equation* is an integral equation which is sometimes misrepresented as a Fredholm's integral equation of the second kind [Kajiya 1986]. In this paper, we confirm suspicions [Arvo 1995] that the *light transport operator* in the rendering equation is *not a Fredholm operator* in the general (common) case. Arvo analyses the specific case of specular transport and explains that specular reflections result in a non-compact operator. We show that the light transport operator is generally non-compact, barring special cases involving specific geometric arrangement and materials. This finding explains why several classes of light-agnostic approximation techniques result in unbounded approximation error.

Numerous theoretical papers served to spring-board the development of practical algorithms. Kajiya [1986] showed that distribution ray tracing [Cook et al. 1984] is a Monte Carlo approximation to the rendering equation. Arvo [1996] derived the operator form of the rendering equation and analyzed the properties of the transport operator for some settings [Arvo 1995]. Similar derivations where recently provided for the volumetric transport equation [Zhang et al. 2019]. Radiosity [Goral et al. 1984] was derived as a finite element method obtained via Galerkin projection of the operator transport equation [Atkinson and Chandler 1996; Heckbert and Winget 1991; Zatz 1993]. Veach [1997] developed the formalism of path tracing and proposed a Markov Chain Monte Carlo estimator. The study of local light transport in frequency space [Durand et al. 2005; Lessig and Fiume 2010; Mahajan et al. 2007; Ramamoorthi and Hanrahan 2004] was useful to target adaptive methods. Monte-Carlo, finite-element and density estimation methods were analyzed in a unified setting using reproducing Kernel bases [Lessig 2012] as a correspondence between continuous functionals and pointwise samples.

Alongside theoretical results, a vast amount of practical knowledge and intuition has been amassed regarding the numerical simulation of light transport. Approaches that employed light-agnostic finite dimensional approximations—any basis such as Fourier, wavelets, radial basis functions—posed significant challenges for high-quality, artifact-free rendering [Gortler et al. 1993; Hanrahan et al. 1991; Sloan et al. 2005]. Galerkin methods that were useful for Lambertian scenes proved difficult to extend to scenes that are not lambertian [Christensen et al. 1996; Immel et al. 1986]. Adaptive computation has evolved to be a key contributor to limiting approximation error in many scenarios such as refinement of radiosity meshes, irradiance or radiance caching [Silvennoinen and Lehtinen 2017], progressive photon mapping [Kaplanyan and Dachsbacher 2013] and baked light maps [Seyb et al. 2020; Silvennoinen and Sloan 2019]. The simplicity and efficacy of Monte Carlo methods has led to their dominance of rendering solutions for offline-rendering and their emerging relevance to real-time rendering. We hypothesize that these seemingly disparate observations stem from a key theoretical property of the transport operator – that *it is not compact.*

Despite tremendous developments it remains a challenge to bound approximation errors, even in simple settings such as purely Lambertian scenes, across arbitrary lighting conditions. In practice, any newly proposed approximation therefore must be validated using a variety of lighting conditions and scenes. We explain these trends from a theoretical perspective, by analyzing properties of the global light transport operator independent of its discretization or any

Siggraph, 0730 0301/2022/1 ART

SIGGRAPH 2022

Very theoretical study of the light transport operator
Few figures and a lot of proofs in the supplemental material

In this presentation, only a subset of the results

Distributions and operators

Radiance distribution:

$$L : \mathcal{S} \to \mathbf{R} =: \mathcal{H}$$

Set of surfaces

We restrict ourselves to diffuse scenes.

Distributions and operators

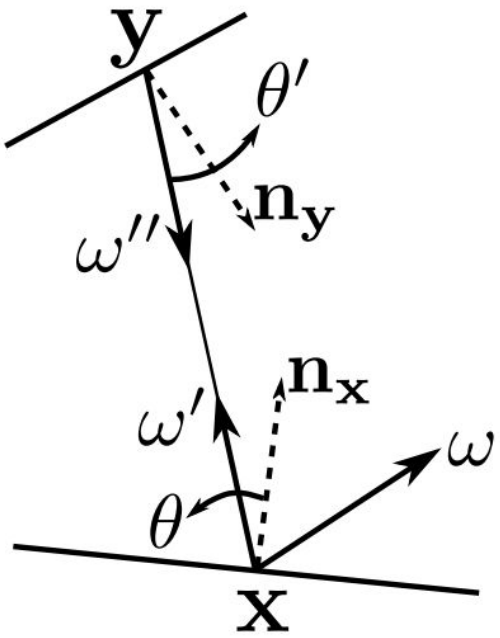
Radiance distribution:

$$L : \mathcal{S} \to \mathbf{R} =: \mathcal{H}$$

Operators:

$$\mathbf{A} : \mathcal{H} \to \mathcal{H}$$

Light transport operator



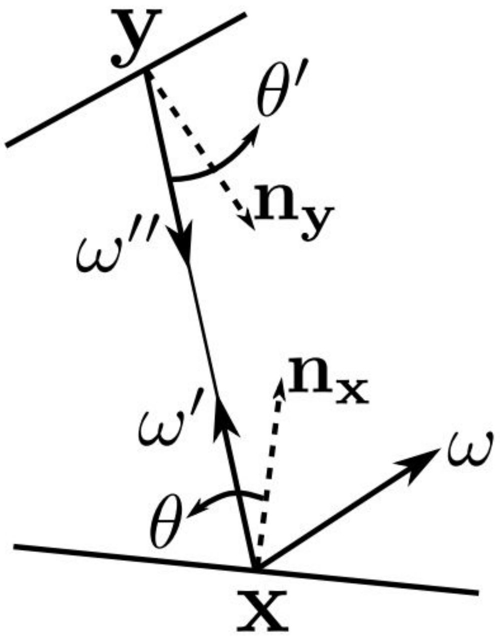
Operator for one bounce of light

$$(\mathbf{T}_b L)(x) = \int_{\mathcal{S}} L(y) v(x,y) \rho(x) \frac{\cos\theta \cos\theta'}{\pi ||x-y||^2} dy$$

visibility

albedo

Light transport operator



Operator for one bounce of light

$\kappa(x, y)$

$$(\mathbf{T}_b L)(x) = \int_{\mathcal{S}} L(y) v(x,y) \rho(x) \frac{\cos\theta \cos\theta'}{\pi ||x-y||^2} dy$$

$$= \int_{\mathcal{S}} L(y) \kappa(x,y) dy$$

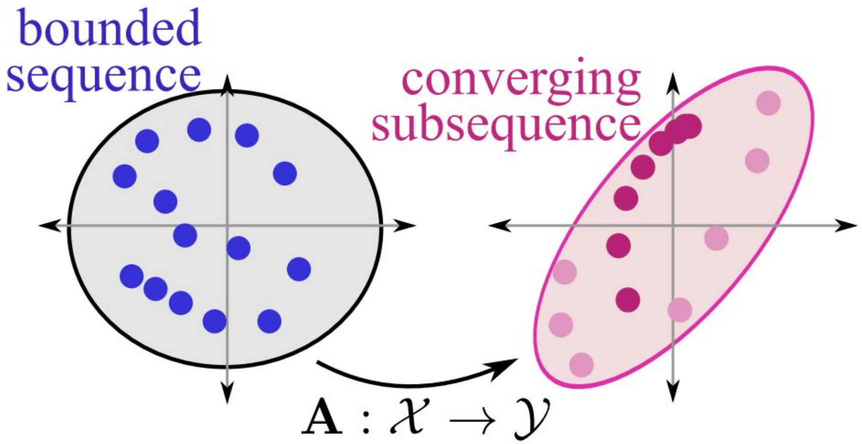
Integral operator, with kernel function

Compact operators

Compact operator:

For all bounded distribution sequence $(L_n)_n$,

$(\mathbf{A}L_n)_n$ has a convergent subsequence.



Compact operators

Compact spaces are spaces where objects can't oscillate indefinitely/arbitrarily

So compact operators can be thought as "smoothing" operators. They take generic (but bounded) sequences, and "compresses" them into sequences that can't oscillate too much (because they have to contain a converging subsequence).

A non-compact operator:

$$\mathbf{A}f = f$$

A compact operator:

$$(\mathbf{A}f)(x) = \int_{\mathbf{R}} G(x-y)f(y)dy$$

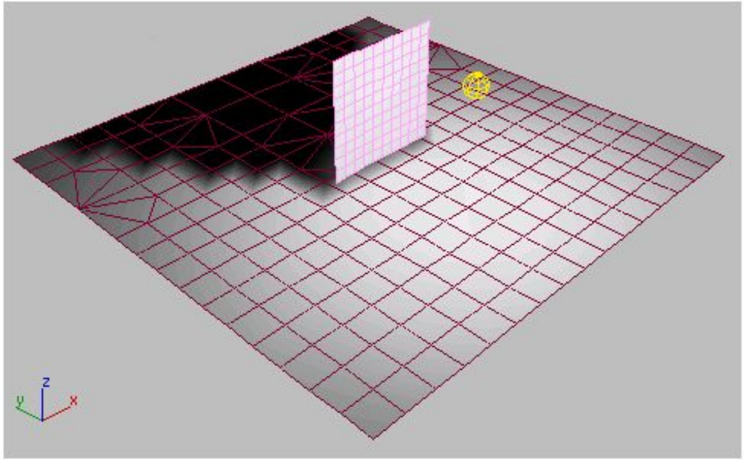
Convolution with a Gaussian kernel: it smoothes the function, blurring high-frequency oscillations. For example, even increasingly oscillating functions, like $f_n(x) = \sin(nx)$ have a converging subsequence (here, it converges to zero).

Is the light transport operator compact ?

Important, because compact operators can be ***uniformly/reliably*** approximated using *finite rank linear operators.*

They can be approximated by their action over finite dimension space, with an error independent of the radiance distribution.

For example, it can be applied to finite element methods, for which we would have an upper bound on the error independent of the radiance distributions.



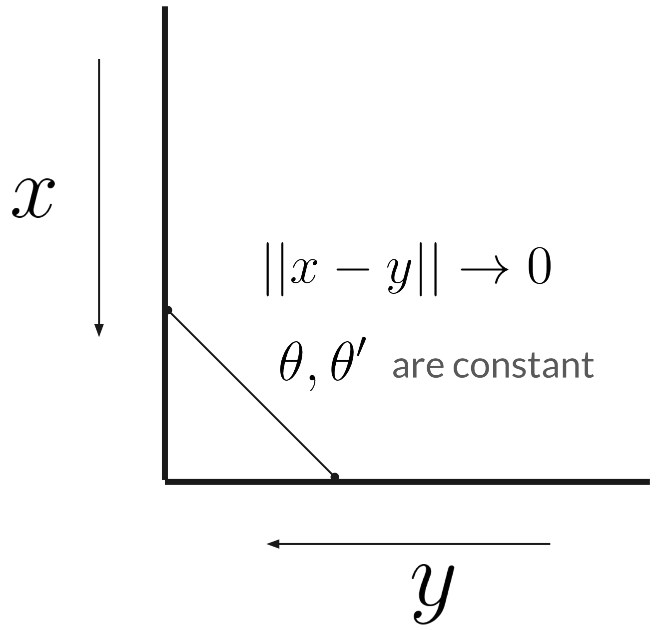
So, is it compact ?

$$\kappa(x,y) = v(x,y)\rho(x)\frac{\cos\theta\cos\theta'}{\pi||x-y||^2}$$

If the kernel is **bounded**, then the light transport operator is **compact.**

Indeed, the integral operator of a square-integrable kernel is called a Hilbert-Schmidt operator, and they are compacts

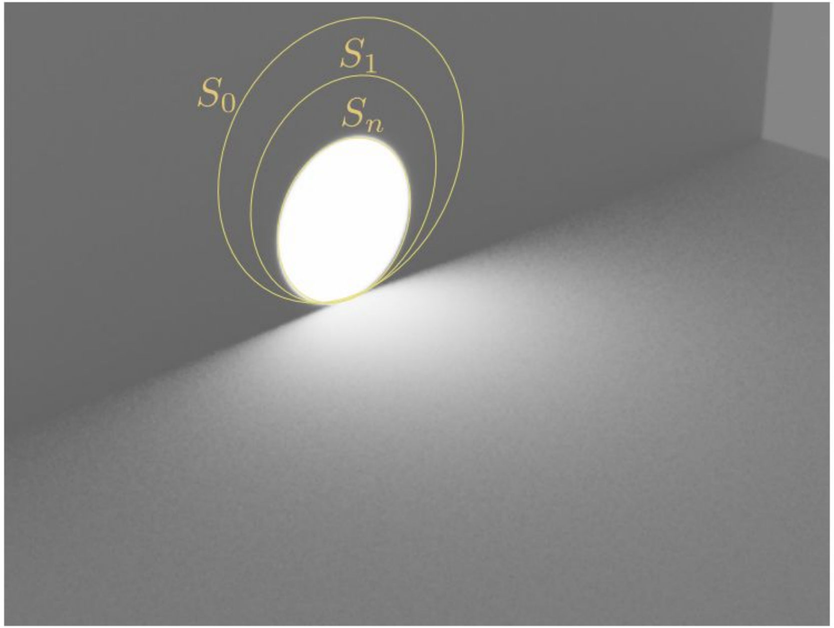

But when is the kernel unbounded ?



$$\kappa(x,y) = v(x,y)\rho(x)\frac{\cos\theta\cos\theta'}{\pi||x-y||^2}$$

Abutting angles break this condition.

Illustration of the non-compactness



Situation:

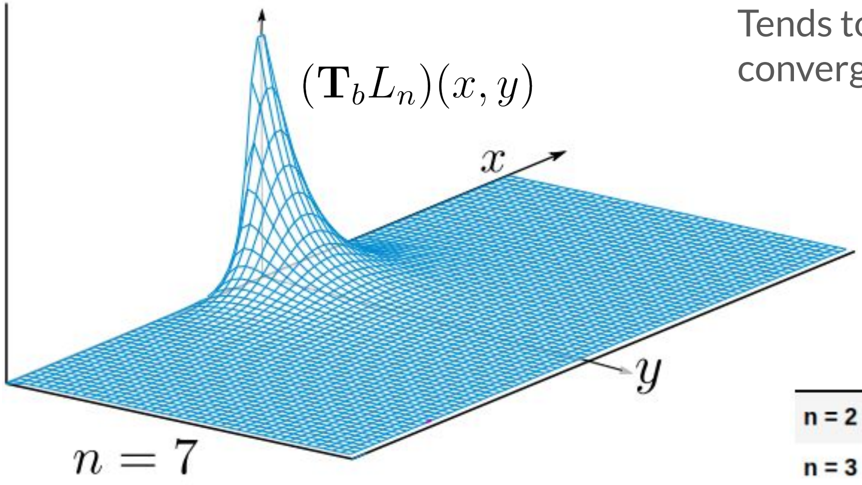
Two abutting planes, and a sequence of circular light sources, such that:

the areas $$a_n = a_0 \cdot \frac{1}{2^n}$$

the emission powers $$e_n = e_0 \cdot 2^n$$

So the initial radiance distributions are bounded in norm.

Illustration of the non-compactness



Tends to a spike/delta distribution -> no (sub-sequence) convergence

The reminders $r_p = ||\mathbf{T}_b L_n - \mathbf{T}_b L_{n+p}||_2$

is increasing and tends to a positive limit.

	p = 1	p = 2	p = 3	p = 4	p = 5	p = 6	p = 7	p = 8
n = 2	0.003146	0.006452	0.009923	0.013522	0.017174	0.020789	0.024277	0.027568
n = 3	0.003371	0.006966	0.010740	0.014606	0.018462	0.022209	0.025769	0.029087
n = 4	0.003658	0.007549	0.011580	0.015638	0.019618	0.023430	0.027013	0.030325
n = 5	0.003950	0.008093	0.012310	0.016488	0.020533	0.024370	0.027950	0.031244
n = 6	0.004199	0.008527	0.012866	0.017112	0.021185	0.025026	0.028595	0.031871
n = 7	0.004385	0.008837	0.013247	0.017528	0.021612	0.025449	0.029007	0.032268
n = 8	0.004510	0.009037	0.013487	0.017785	0.021871	0.025703	0.029253	0.032504

Another result: operator decomposition

The article presents another result, on the general (non-diffuse) light transport operator **T**.

If the scene has a finite number of materials $m_1, \ldots, m_n$

$$\mathbf{T}L = \sum_{i=1}^{n} \sum_{j>0} \rho_{ij} \langle \phi_j^i, L \rangle \psi_j^i$$

where

$\{\rho_{ij}\}_j$ are the eigenvalues of material i

$\{\psi_j^i\}_j$ are essentially "eigenfunctions of material i"

$\{\phi_j^i\}_j$ are the images of $\{\psi_j^i\}_j$ after light exchange

Implications

Compactness:

Mainly of theoretical result, to study existing finite-rank / finite-bases methods. The result is that in most cases, the error using such methods is unbounded. They argue that better understanding of this operator can guide the choice between these methods.

Decomposition:

They don't elaborate much on this result, but they say that explain that eigenvalues/eigenfunctions are hard to estimate in this case. So also mainly a theoretical result too.

Thank you, questions ?