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K-SVD: An Algorithm for Designing Overcomplete Dictionaries for Sparse Representation

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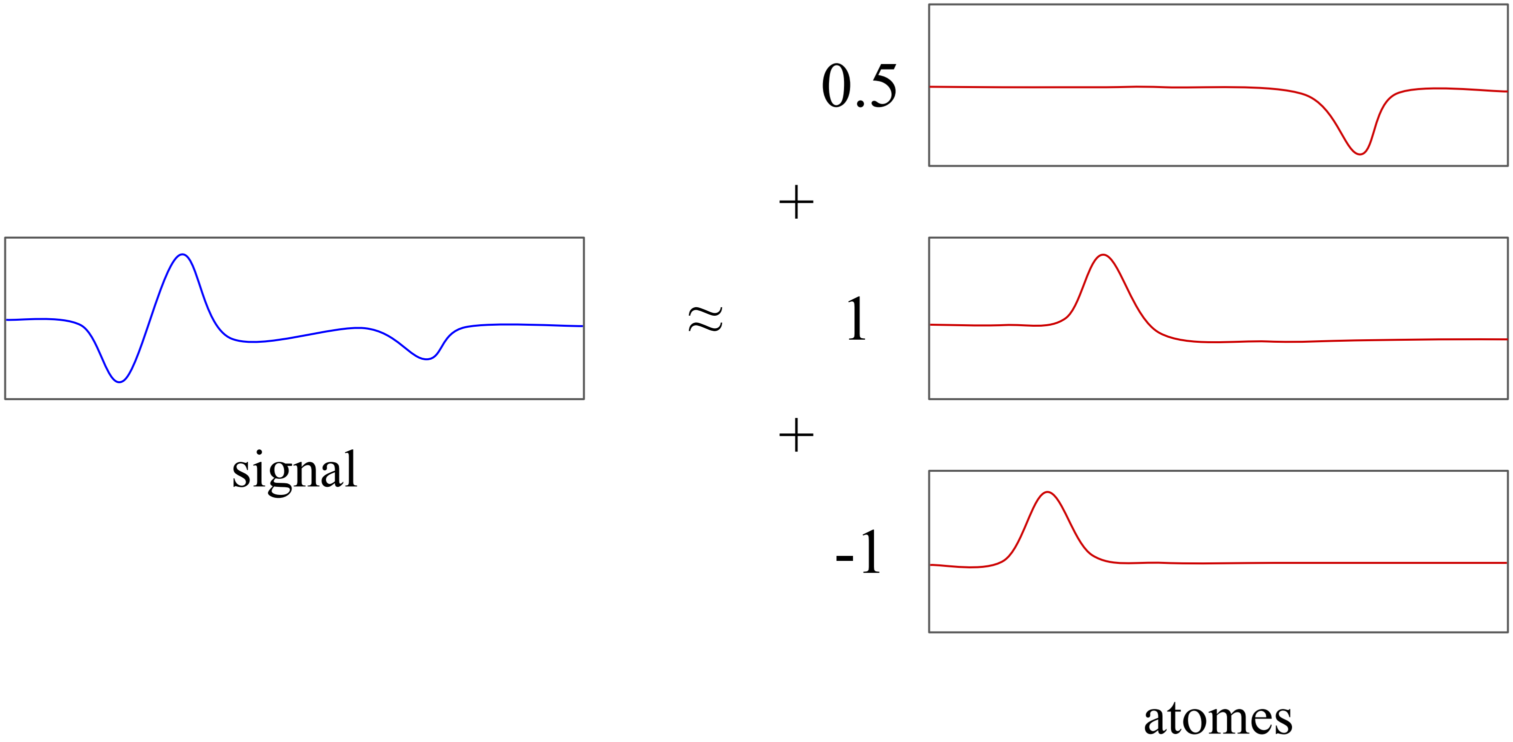
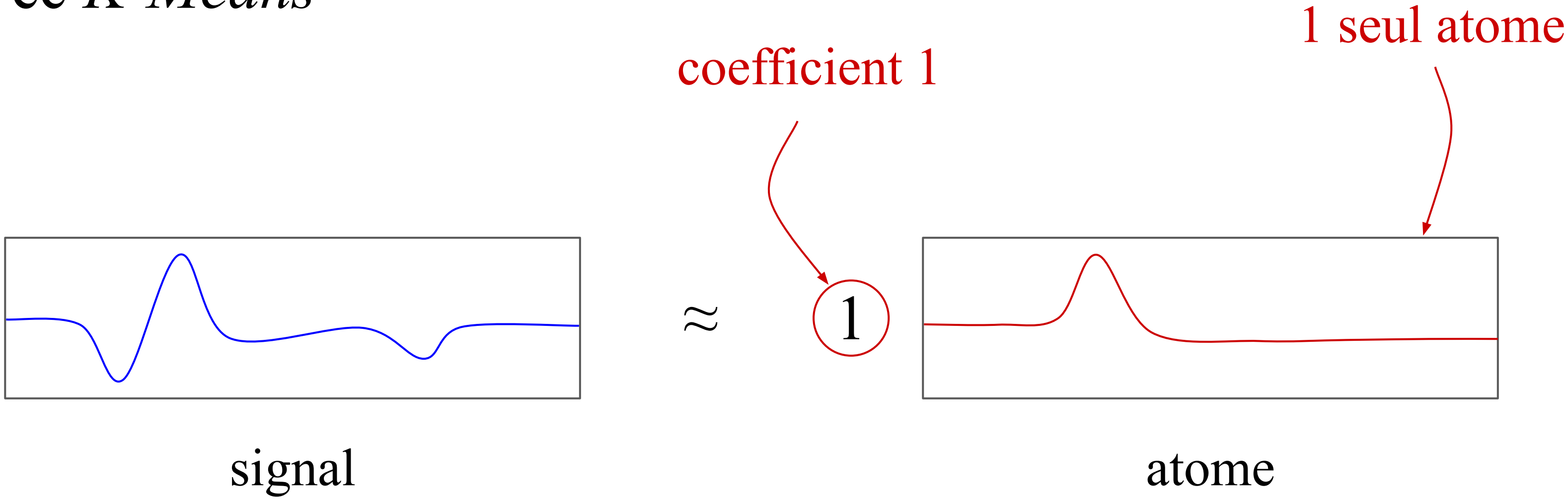
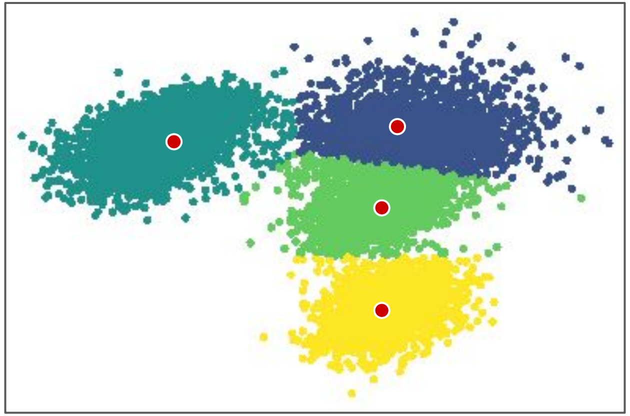
Représentation sparse

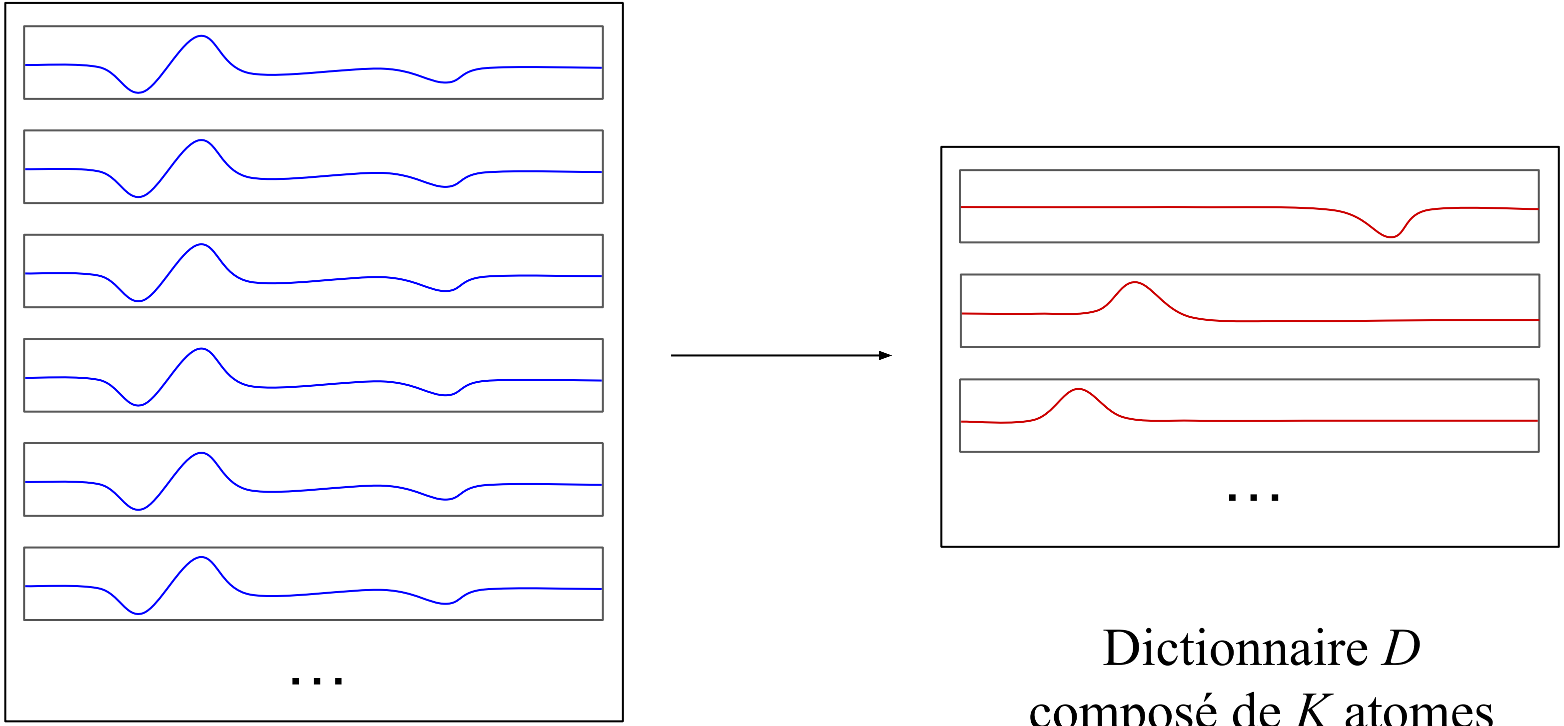


Lien avec *K-Means*





K-SVD et *K-Means*



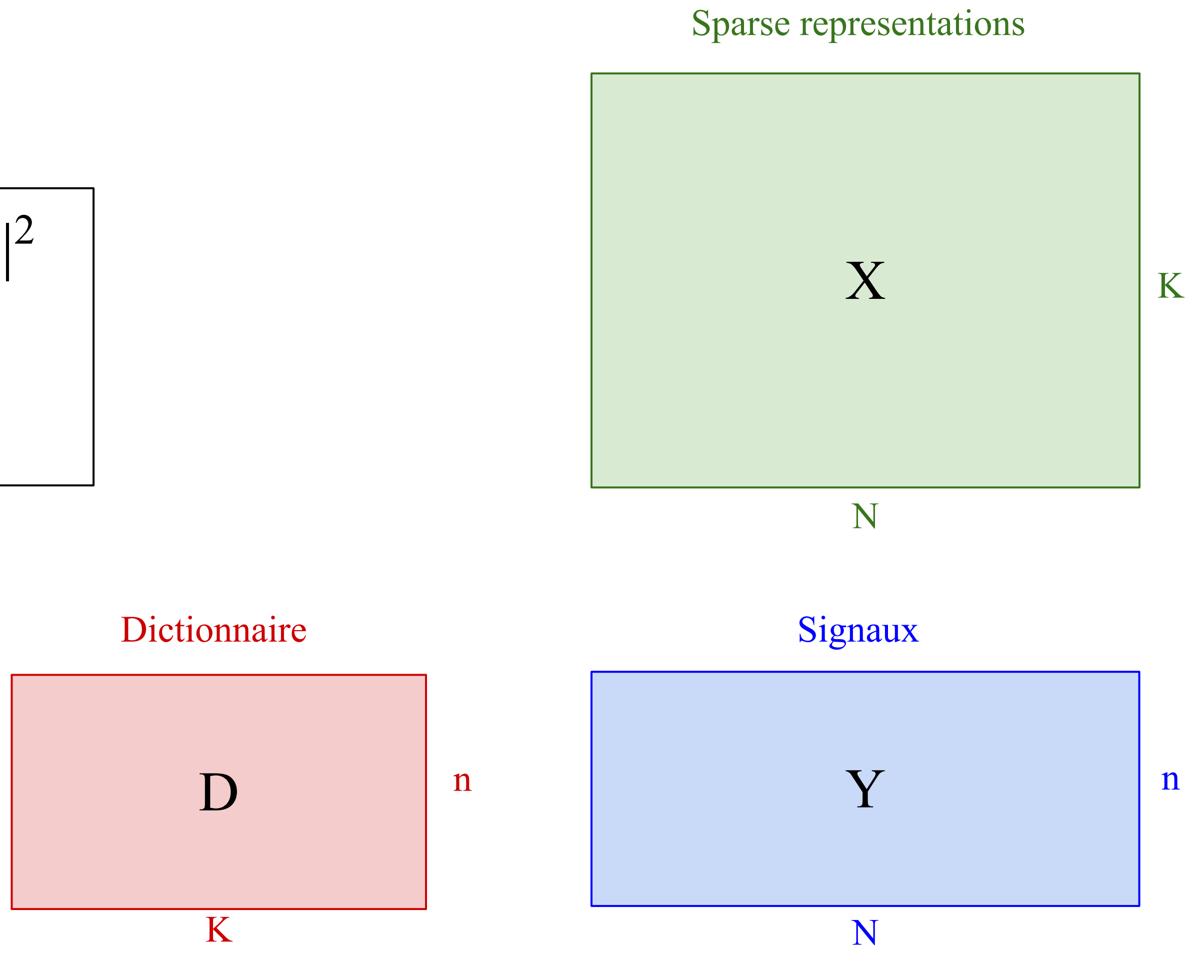
N signaux

Dictionnaire *D*
composé de *K* atomes

K-Means

$$\min_{D,\,X} \|Y - DX\|^2$$

$$\text{avec} \quad x_i = e_j$$



K-Means

$$\min_{D,\,X} \|Y - DX\|^2$$

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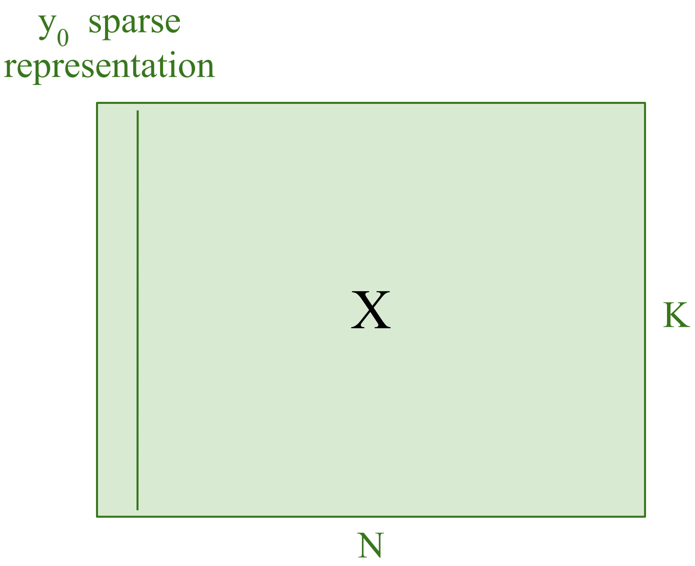
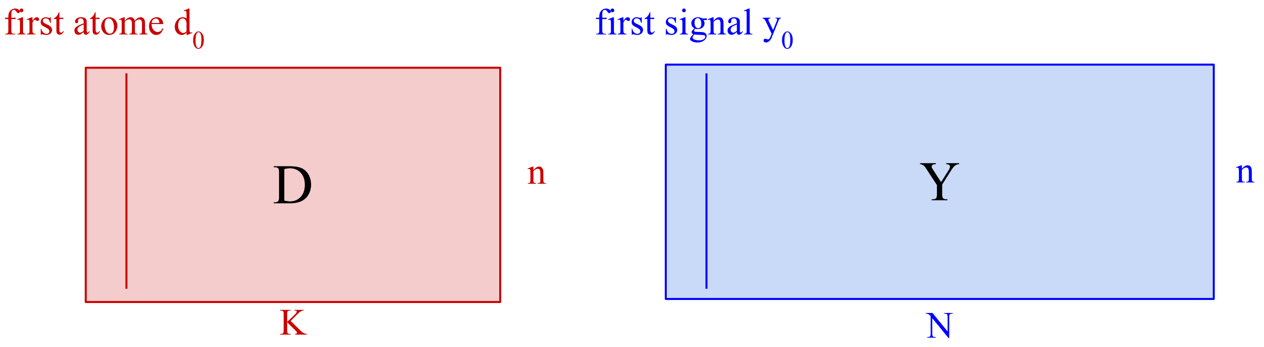




K-Means

$$\boxed{\begin{gathered}\min_{D,\,X} \|Y - DX\|^2 \\[1em] \text{avec} \quad x_i = e_j\end{gathered}}$$

```
// K-Means
while (not converged)

     // Sparse Coding
     Optimize X with D fixed

     // Dictionary Update
     Optimize D with X fixed
```

K-Means

$$\min_{D, X} \|Y - DX\|^2$$

$$\text{avec} \quad x_i = e_j$$

```
// K-Means
while (not converged)

    // Sparse Coding
    Optimize X with D fixed

    // Dictionary Update
    Optimize D with X fixed
```

Chaque x_i est représenté par l'atome d_j le plus proche

K-Means

$$\min_{D, X} \|Y - DX\|^2$$

$$\text{avec} \quad x_i = e_j$$

```
// K-Means
while (not converged)

    // Sparse Coding
    Optimize X with D fixed

    // Dictionary Update
    Optimize D with X fixed
```

Chaque d_i est le barycentre de tous les x_i qui l'utilisent comme représentation

De *K-Means* à *K-SVD*

$$\min_{D,X} \|Y - DX\|^2$$

avec $x_i = e_j$

$$\min_{D,X} \|Y - DX\|^2$$

$\boxed{\text{avec } x_i \text{ sparse}}$

```
// K-Means
while (not converged)

    // Sparse Coding
    Optimize X with D fixed

    // Dictionary Update
    Optimize D with X fixed
```

```
// K-SVD
while (not converged)

  ? // Sparse Coding
    Optimize X with D fixed

  ? // Dictionary Update
    Optimize D with X fixed
```

K-SVD

$$\boxed{\begin{array}{c}\min_{D,\,X} \|Y - DX\|^2 \\[1em] \text{avec } x_i \text{ sparse}\end{array}}$$

```
// K-SVD
while (not converged)

    // Sparse Coding
    Optimize X with D fixed

?   // Dictionary Update
    Optimize D with X fixed
```

$\min \|x_i\|_0$ avec $\|y_i - Dx_i\|^2 \leq \epsilon$

ou

$\min \|y_i - Dx_i\|^2$ avec $\|x_i\|_0 \leq T_0$

→ NP-complet → Algorithmes de poursuite

Matching Pursuit (MP)

```
// Matching Pursuit
x_i = {0 ... 0}
r = y_i
while (not converged)

    k  = max_j <r, d_j>

    x_i(k) = <r, d_k>

    r  = y - Σ_j x_i(j).d_j
```

Condition sur $||x||_0$ et $||y - Dx||^2$

MP → Orthogonal Matching Pursuit (OMP)

```
// Matching Pursuit
x_i = {0 ... 0}
r = y_i
while (not converged)

    k = max_j <r, d_j>

    x_i(k) = <r, d_k>

    r = y_i - Σ_j x_i(j).d_j
```

```
// Orthogonal Matching Pursuit
x_i = {0 ... 0}
L = {}
r = y_i
while (not converged)

    k = max_j <r, d_j>

    L += {d_k}

    r = y_i - Decomposition of y_i on
L
```

K-SVD

$$\boxed{\min_{D,\,X} \|Y - DX\|^2 \quad \text{avec } x_i \text{ sparse}}$$

OMP

```
// K-SVD
while (not converged)

    // Sparse Coding
    Optimize X with D fixed

    // Dictionary Update
    Optimize D with X fixed
```

Comme K-Means : ajuster chaque atome par rapport aux signaux qui l'utilisent.

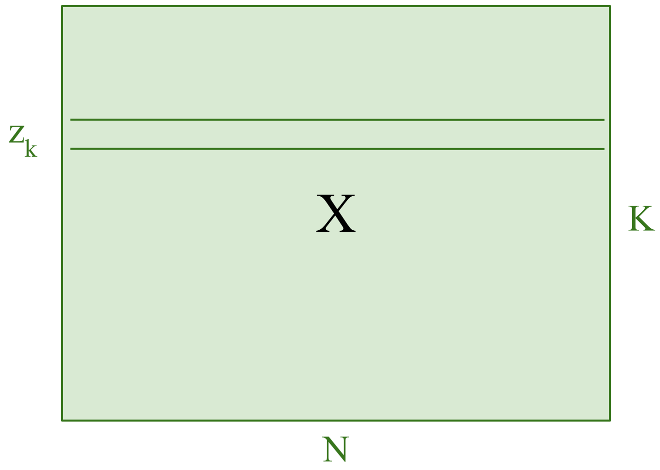
A chaque fois, on minimise $\|Y - DX\|^2$ en fixant toutes les données, sauf l'atome d_k et les coefficients associés $z_k = (X^T)_k$.

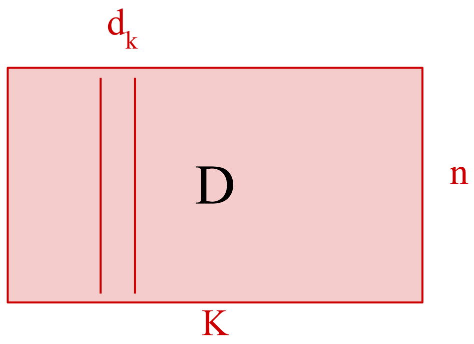
K-SVD

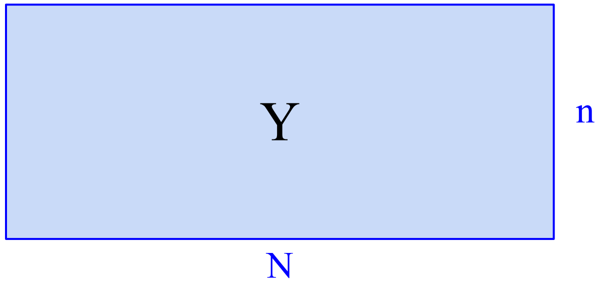
$$\|Y - DX\|^2 = \|Y - \sum_{j=1}^{K} d_j z_j\|^2$$

$$= \|\Big(Y - \sum_{j \neq k}^{K} d_j z_j\Big) - d_k z_k\|^2$$

$$= \|E_k - d_k z_k\|^2$$







K-SVD

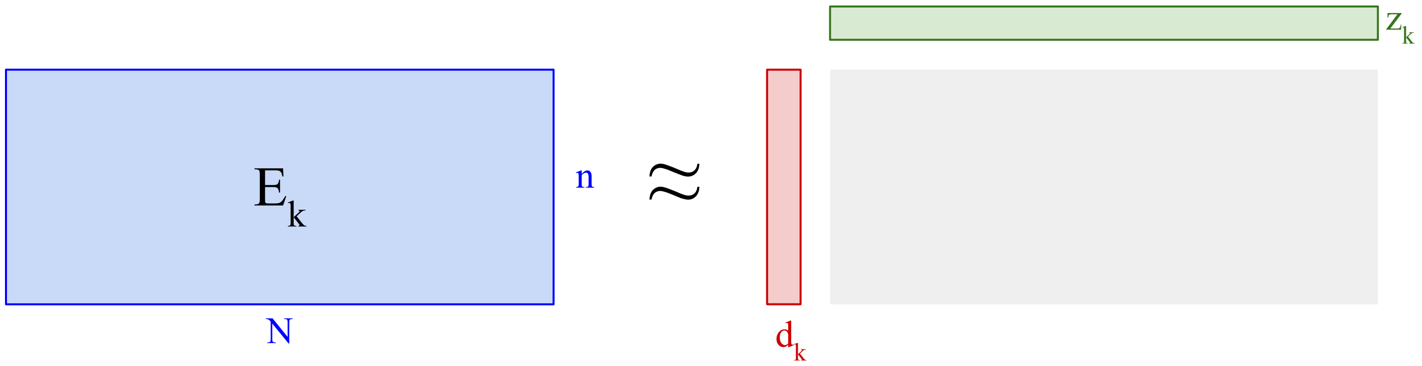
$$\|Y - DX\|^2 = \|Y - \sum_{j=1}^{K} d_j z_j\|^2$$

$$= \|\Big(Y - \sum_{j \neq k}^{K} d_j z_j\Big) - d_k z_k\|^2$$

$$= \|E_k - d_k z_k\|^2$$

On isole l'atome et les coefficients impliqués

Approximation de rang 1



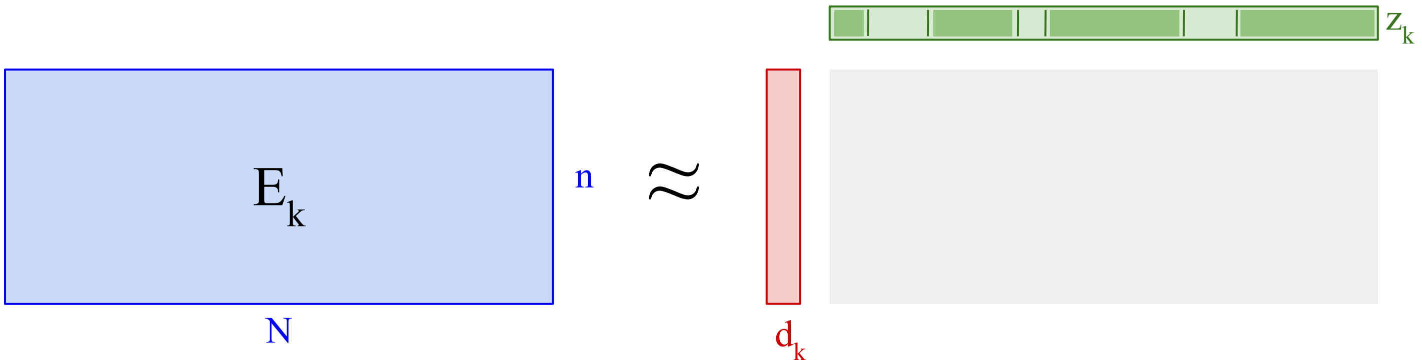
K-SVD

$$\|Y - DX\|^2 = \|Y - \sum_{j=1}^{K} d_j z_j\|^2$$

$$= \|\left(Y - \sum_{j \neq k}^{K} d_j z_j\right) - d_k z_k\|^2$$

$$= \|E_k - d_k z_k\|^2$$

Z est sparse !

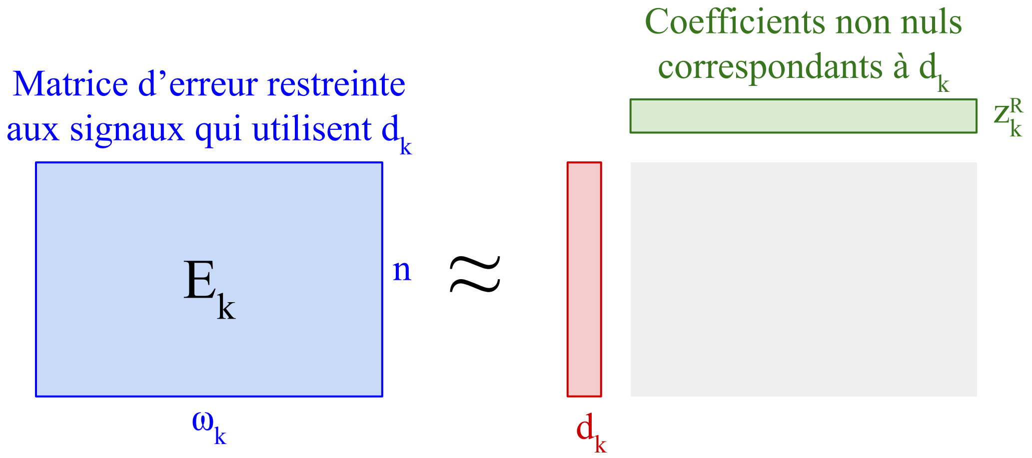


K-SVD

$$\|Y - DX\|^2 = \|Y - \sum_{j=1}^{K} d_j z_j\|^2$$

$$= \|\left(Y - \sum_{j \neq k}^{K} d_j z_j\right) - d_k z_k\|^2$$

$$= \|E_k - d_k z_k\|^2$$



K-SVD - Convergence

$$\min_{D, X} ||Y - DX||^2$$

$$\text{avec } x_i \text{ sparse}$$

```
// K-SVD
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    // Dictionary Update
    Optimize D with X fixed
```

OMP

Rank 1 Approx

Mise à jour du dictionnaire :
résultat exact, l'erreur décroît

Mise à jour des coefficients :
algorithmes de poursuite, approximations …
(en pratique, les résultats sont très bons quand $T_0 << n$)

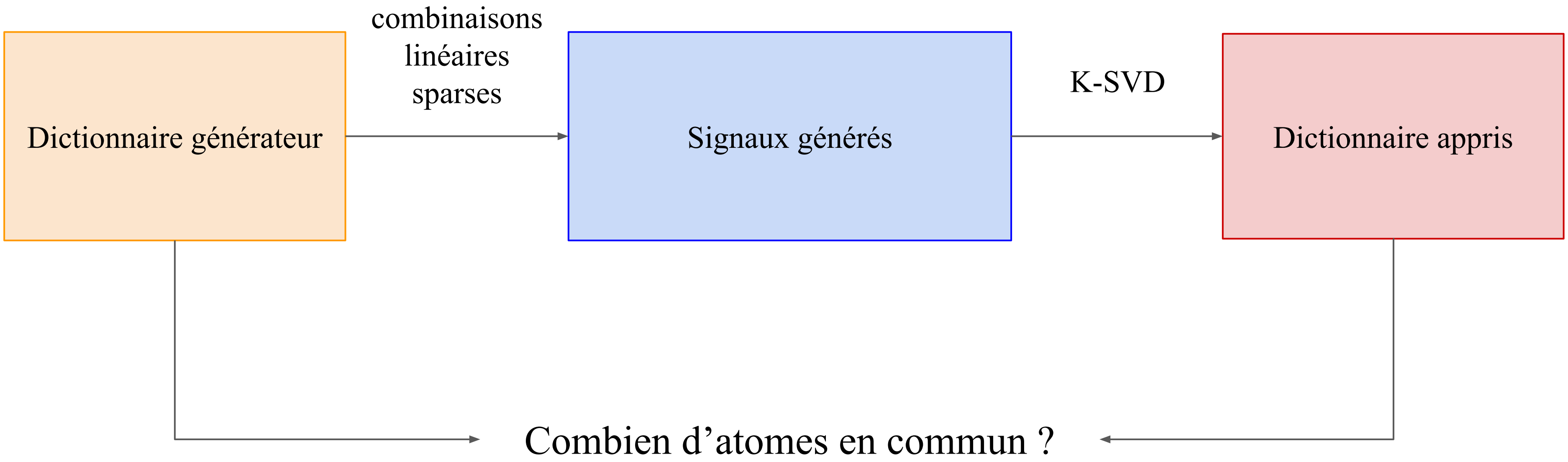
K-SVD - Améliorations

Atome constant :
L'atome 0 du dictionnaire est constant, il sera choisi dans la décomposition de tous les signaux pour représenter la composante continue.

Atome inutile :
Quand un atome du dictionnaire est très peu utilisé, il peut être remplacé par le signal le moins bien représenté.

Atomes en double :
Quand deux atomes du dictionnaire sont trop proches, un des deux peut être remplacé par le signal le moins bien représenté.

K-SVD - Validation

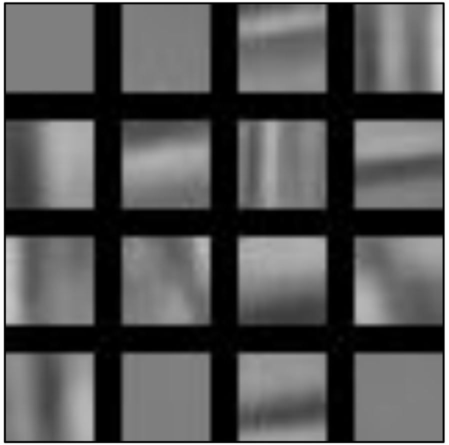


K-SVD

Originale

Approximation sparse

16 atomes
5-sparsity



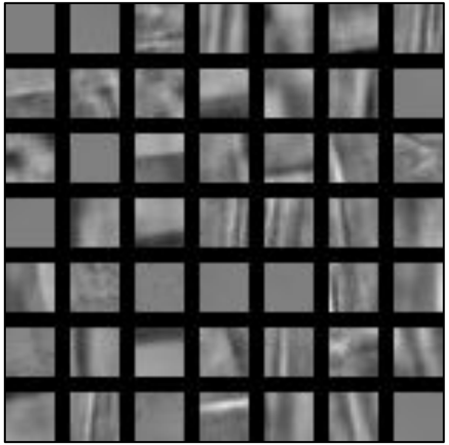
Dictionnaire

K-SVD

Originale

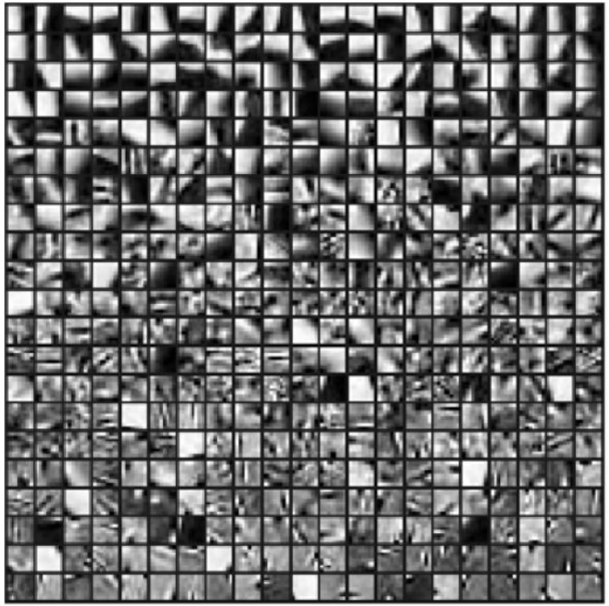
Approximation sparse

49 atomes
10-sparsity

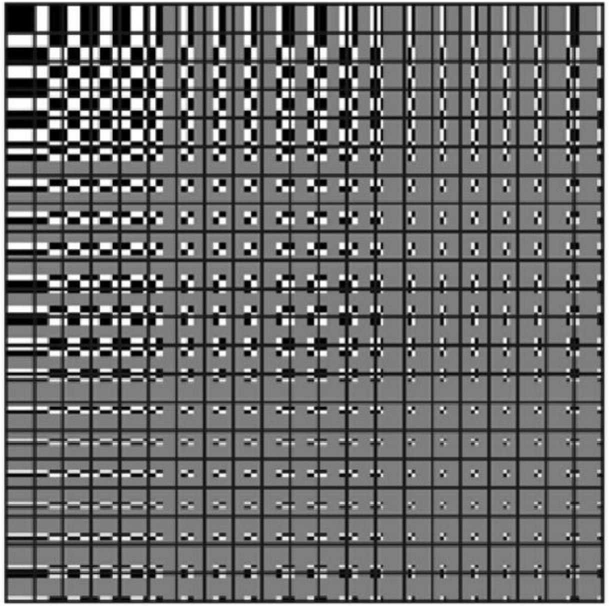


Dictionnaire

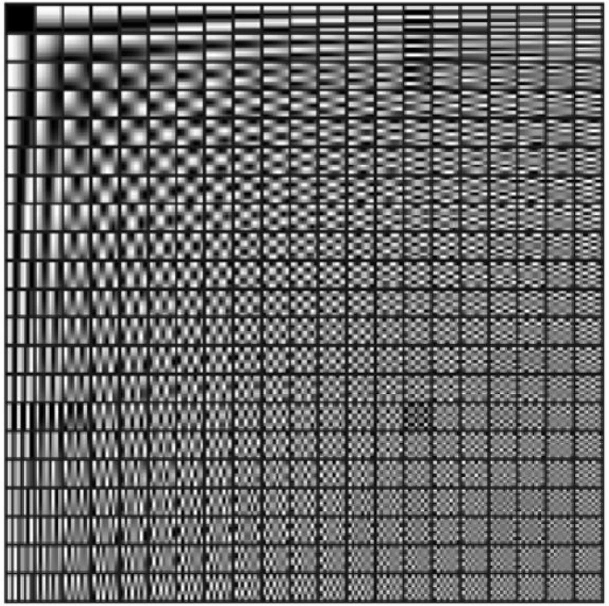
K-SVD



K-SVD



Haar



DCT

Le K-SVD original

2006 - Aharon et al. - **K-SVD: An Algorithm for Designing Overcomplete Dictionaries for Sparse Representation** - Transactions on signal processing

Une analyse détaillée du K-SVD

2012 - Lebrun et Leclaire - **An Implementation and Detailed Analysis of the K-SVD Image Denoising Algorithm** - Image Processing On Line
http://www.ipol.im/pub/art/2012/llm-ksvd/

Des comparaisons sur les méthodes de poursuite

1999 - Cotter et al. - **Forward sequential algorithms for best basis selection** - IEE Proceedings: Vision, Image and Signal Processing

K-SVD - Applications

Compression avec pertes → Débruitage

http://www.ipol.im/pub/art/2012/llm-ksvd/