It's a small step for man, but...

Large Steps in Inverse Rendering of Geometry

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Differentiable rendering, again

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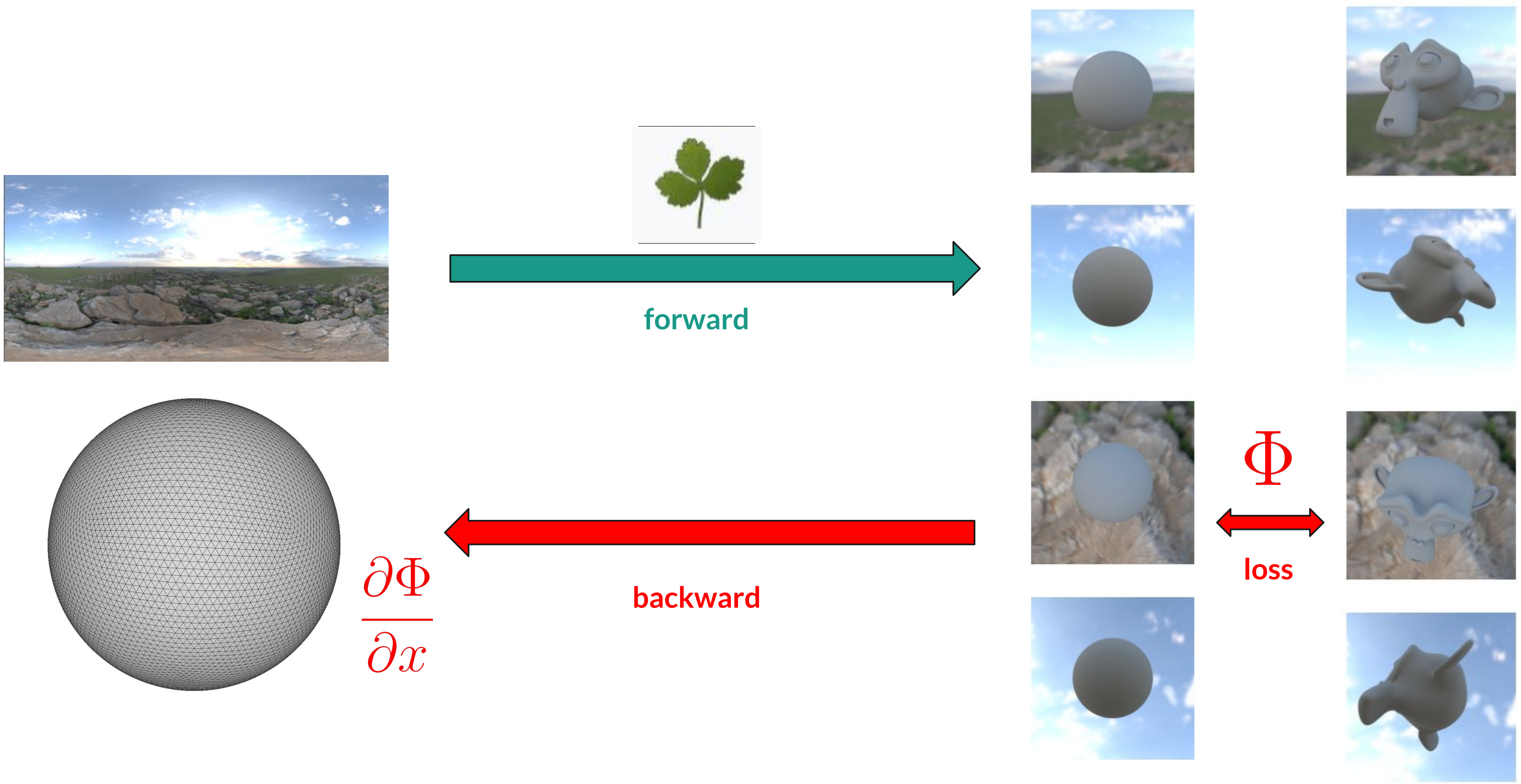


Gradient descent

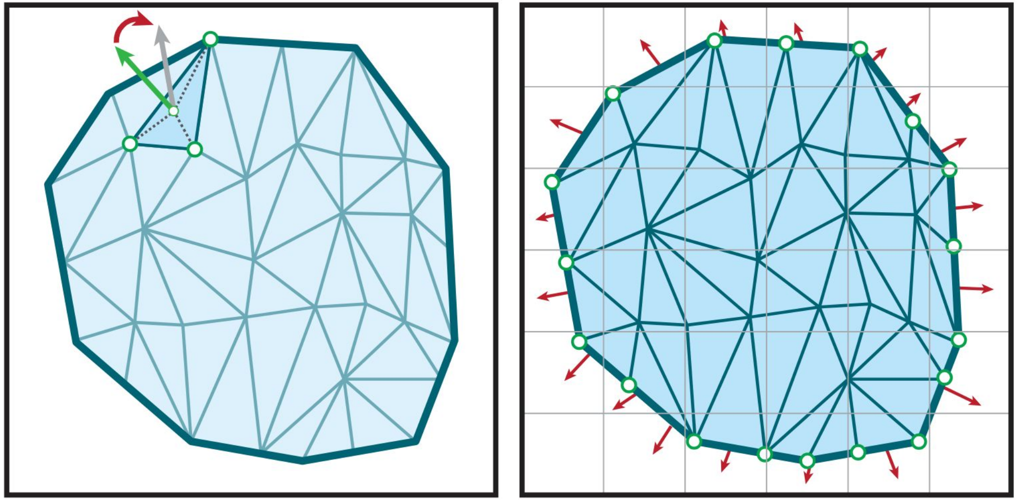
Goal: $\underset{\mathbf{x} \in \mathbb{R}^{n \times 3}}{\text{minimize}} \quad \Phi(R(\mathbf{x}))$

GD step: $\mathbf{x} \leftarrow \mathbf{x} - \eta \frac{\partial \Phi}{\partial \mathbf{x}}$

Φ: loss function
R: rendering function
x: vertex positions

η: learning rate

Gradients w/r to positions



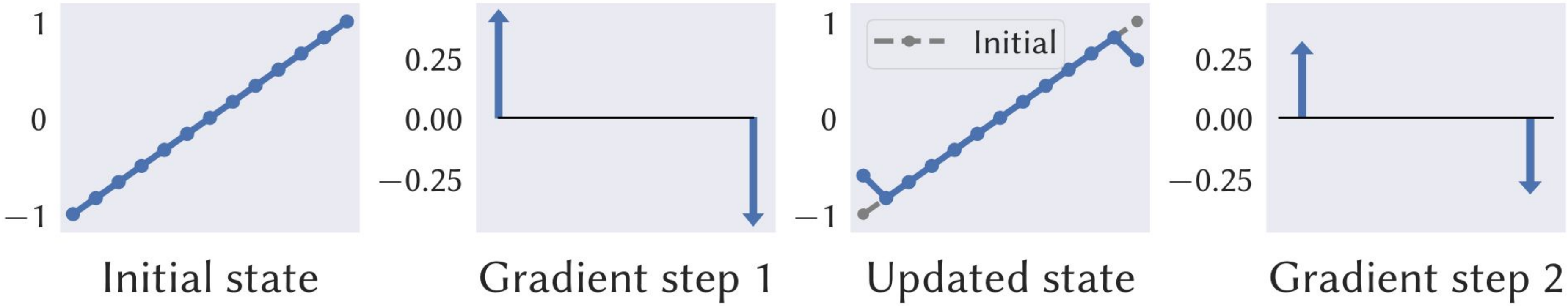
(a) Shading gradients

(b) Silhouette gradients

Smooth, small in magnitude

Sparse, very large in magnitude

In theory, why it doesn't work



Toy 1D "mesh" example:

We start from a linear progression in [-1, 1], and target a linear progression from [-0.5, 0.5].
We assume that we only have access to **sparse "silhouette" gradients**.

Result: within a few iterations, we'll have a **tangled shape, with multiple inverted elements**.

Indeed, it doesn't work

Sparse gradients

+

Connectivity ignored

Laplacian operator and Dirichlet energy

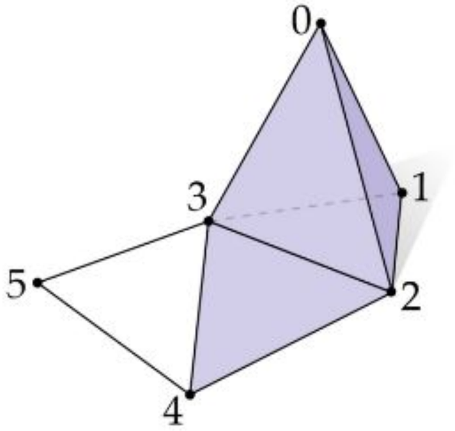
Laplacian operator on a scalar field:

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$$

Dirichlet energy:

$$E(f) := \frac{1}{2} \int_\Omega \|\nabla f\|^2 \, \mathrm{d}\mathbf{x} = C - \frac{1}{2} \langle f, \Delta f \rangle$$

Discrete Laplacian operator

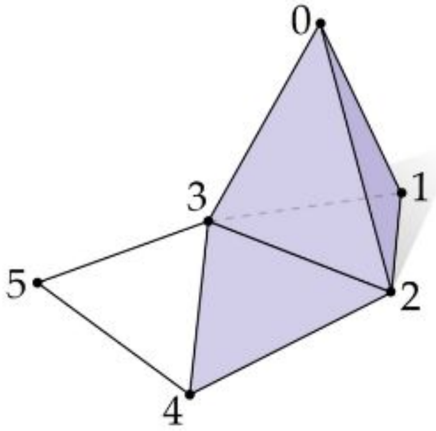


$$\mathbf{L}_{ij} = \begin{bmatrix} -\sum & w_{01} & w_{02} & w_{03} & 0 & 0 \\ w_{10} & -\sum & w_{12} & w_{13} & 0 & 0 \\ w_{20} & w_{21} & -\sum & w_{23} & w_{24} & 0 \\ w_{30} & w_{31} & w_{32} & -\sum & w_{34} & w_{35} \\ 0 & 0 & w_{42} & w_{43} & -\sum & w_{45} \\ 0 & 0 & 0 & w_{53} & w_{54} & -\sum \end{bmatrix}$$

Combinatorial Laplacian operator: the (non-zero) weights are equal to 1

(there are other usual definitions, but this one is sufficient here)

Discrete Laplacian operator



$$\mathbf{L}_{ij} = \begin{bmatrix} -\sum & w_{01} & w_{02} & w_{03} & 0 & 0 \\ w_{10} & -\sum & w_{12} & w_{13} & 0 & 0 \\ w_{20} & w_{21} & -\sum & w_{23} & w_{24} & 0 \\ w_{30} & w_{31} & w_{32} & -\sum & w_{34} & w_{35} \\ 0 & 0 & w_{42} & w_{43} & -\sum & w_{45} \\ 0 & 0 & 0 & w_{53} & w_{54} & -\sum \end{bmatrix}$$

L is a *sparse* operator.

We can also define the Dirichlet energy in terms of **L**:

$$E(\mathbf{f}) = \frac{1}{2}\langle \mathbf{f}, \mathbf{L}\mathbf{f} \rangle = \frac{1}{2}\mathbf{f}^T \mathbf{L}\mathbf{f}$$

Common solution - Loss regularization

New objective:

$$\min_{\mathbf{x}\in\Omega} \underbrace{\boldsymbol{\Phi}(\mathbf{x})}_{\text{"True" objective}} + \underbrace{\lambda \mathbf{x}^{\top} \mathbf{L}\, \mathbf{x}}_{\text{Regularization term}}$$

New gradient descent step:

$$\mathbf{x} \leftarrow \mathbf{x} - \eta \left(\frac{\partial \Phi}{\partial \mathbf{x}} + \lambda \mathbf{L}\mathbf{x} \right)$$

Common solution - Loss regularization -> not magic

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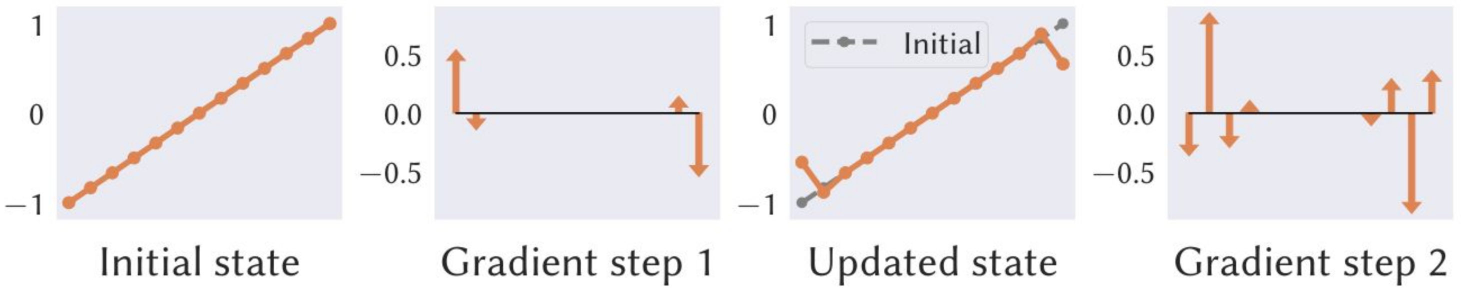
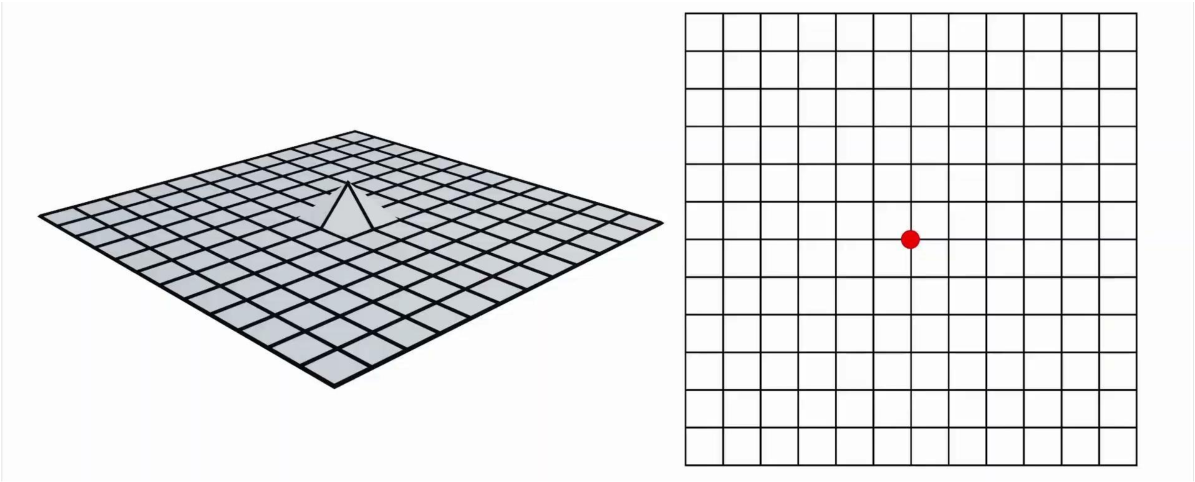
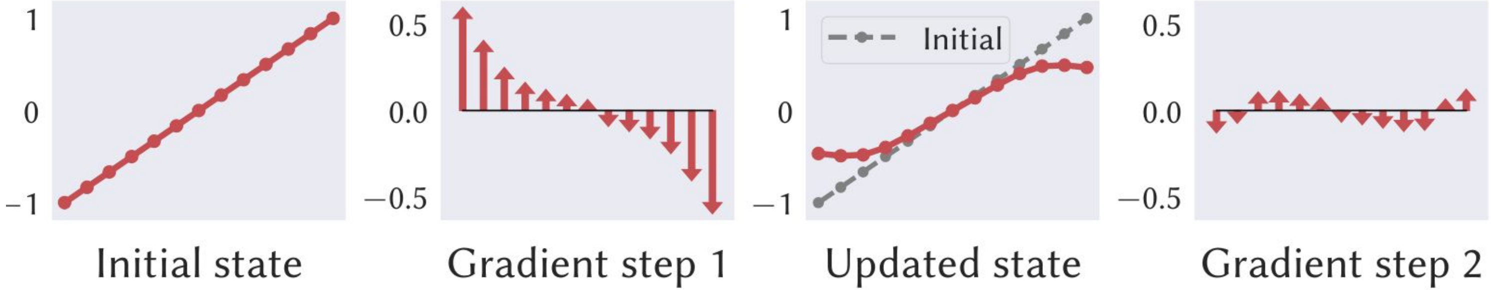




Second-order - Newton's method

Equivalent optimisation step:

$$\mathbf{x} \leftarrow \mathbf{x} - \eta \left(\frac{\partial^2 \Phi}{\partial \mathbf{x}^2} + \lambda \mathbf{L} \right)^{-1} \left(\frac{\partial \Phi}{\partial \mathbf{x}} + \lambda \mathbf{L} \mathbf{x} \right)$$

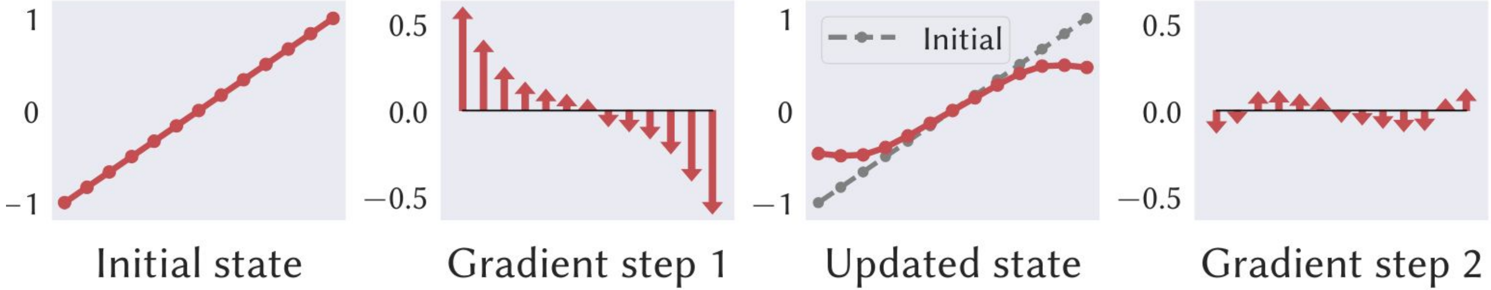


Second-order - Newton's method

Equivalent optimisation step: $$\mathbf{x} \leftarrow \mathbf{x} - \eta \boxed{\left(\frac{\partial^2 \Phi}{\partial \mathbf{x}^2} + \lambda \mathbf{L}\right)^{-1}} \left(\frac{\partial \Phi}{\partial \mathbf{x}} + \lambda \mathbf{L}\mathbf{x}\right)$$

but no hope of having access to that

propagates gradient updates to the whole mesh



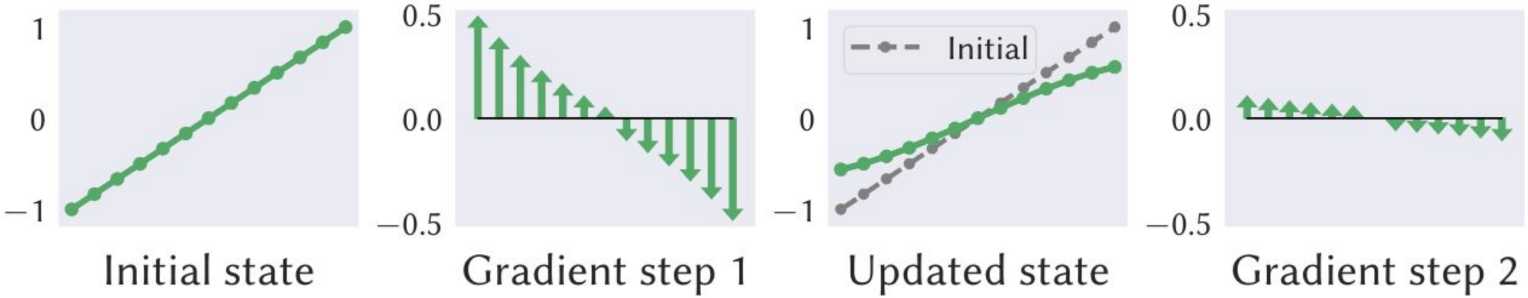
Their method

p=1,2,...

Optimisation step:

$$\mathbf{x} \leftarrow \mathbf{x} - \eta \boxed{(\mathbf{I} + \lambda \mathbf{L})^{-p}} \frac{\partial \Phi}{\partial \mathbf{x}}$$

L is a sparse operator, so can be solved efficiently



Their method - interpretation

Related to the heat diffusion equation.
Given an initial heat distribution **u**, the diffused heat distribution **x** after a time interval λ is the solution of:

$$\underset{\mathbf{x}}{\operatorname{argmin}}\ \frac{1}{2}\|\mathbf{x}-\mathbf{u}\|^2+\lambda\frac{1}{2}\operatorname{tr}\left(\mathbf{x}^\top\mathbf{L}\mathbf{x}\right)$$

It is given by (Euler-Lagrange equation):

$$\mathbf{x}=\underbrace{(\mathbf{I}+\lambda\mathbf{L})^{-1}}_{\text{diffusion operator}}\mathbf{u}$$

diffusion operator

Related to the general formula: $(\mathbf{I}-\mathbf{A})^{-1}=\sum_{k\geq 0}\mathbf{A}^k$

Their method - interpretation

Back to our minimization problem. If we decide to express **x** as the diffusion of some latent variable **u**, i.e. we want to minimize:

$$\Phi(\mathbf{x}(\mathbf{u}))$$

So the gradient descent step becomes:

$$\mathbf{u} \leftarrow \mathbf{u} - \eta \frac{\partial \mathbf{x}}{\partial \mathbf{u}} \frac{\partial \Phi}{\partial \mathbf{x}}$$

Using $\mathbf{x} = (\mathbf{I} + \lambda \mathbf{L})^{-1} \mathbf{u}$

the gradient descent step becomes:

$$\mathbf{x} \leftarrow (\mathbf{I} + \lambda \mathbf{L})^{-1} \left(\mathbf{u} - \eta \frac{\partial \mathbf{x}}{\partial \mathbf{u}} \frac{\partial \Phi}{\partial \mathbf{x}}\right) = \mathbf{x} - \eta \, (\mathbf{I} + \lambda \mathbf{L})^{-2} \frac{\partial \Phi}{\partial \mathbf{x}}$$

which is the optimization step of their method (with p=2).

Optimization scheme

Gradient descent with (first-order) momentum

$$\mathbf{g} \leftarrow (\mathbf{I} + \lambda \mathbf{L})^{-p} \frac{\partial \Phi}{\partial \mathbf{x}},$$
$$\mathbf{m}_1 \leftarrow \beta_1 \mathbf{m}_1 + (1 - \beta_1)\mathbf{g}$$
$$\mathbf{u} \leftarrow \mathbf{u} - \eta \frac{\mathbf{m}_1}{1 - \beta_1^k}$$

Adam optimizer

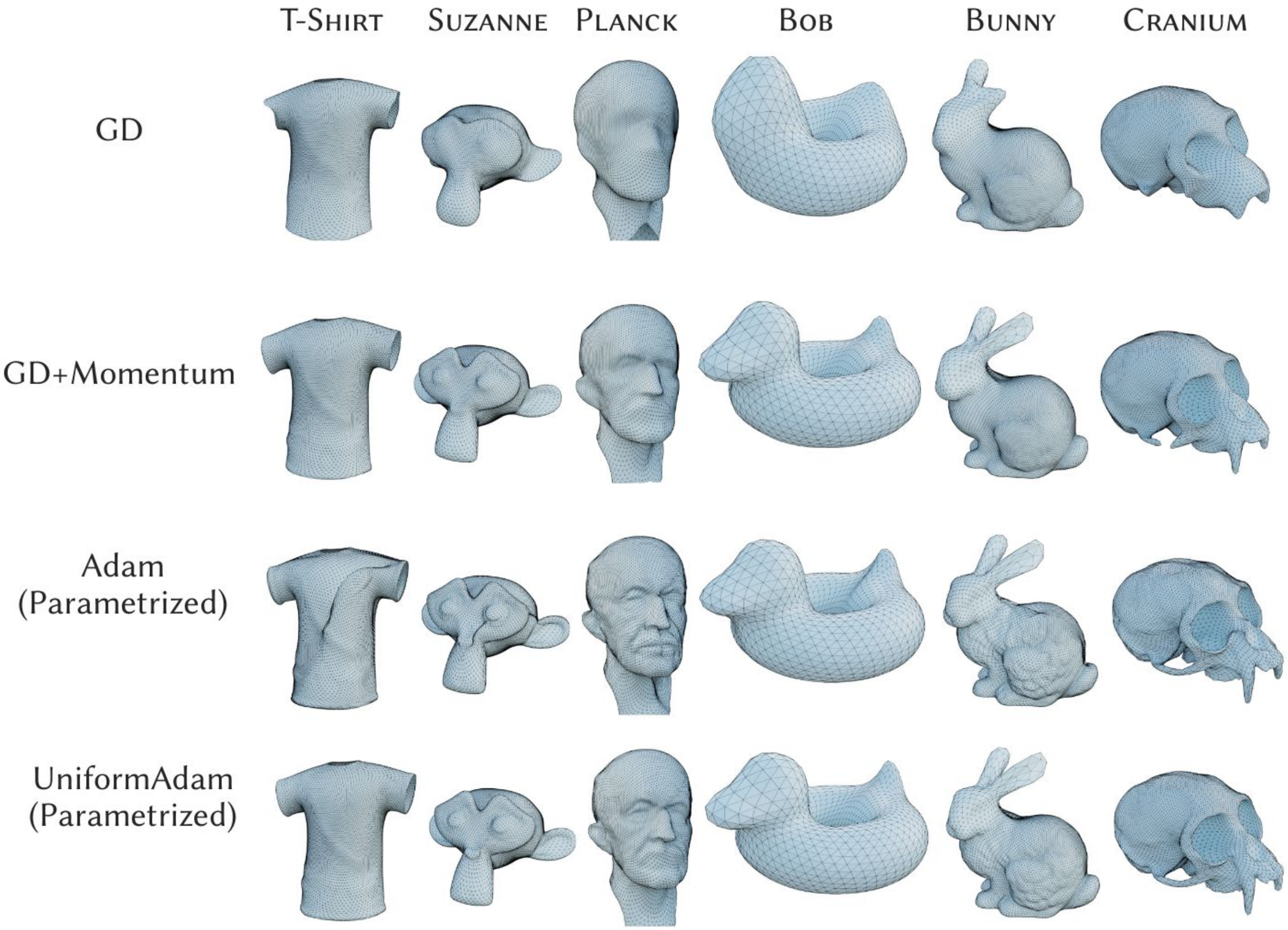
$$\mathbf{g} \leftarrow (\mathbf{I} + \lambda \mathbf{L})^{-p} \frac{\partial \Phi}{\partial \mathbf{x}},$$
$$\mathbf{m}_1 \leftarrow \beta_1 \mathbf{m}_1 + (1 - \beta_1)\mathbf{g}$$
$$\mathbf{m}_2 \leftarrow \beta_2 \mathbf{m}_2 + (1 - \beta_2)\mathbf{g}^2$$
$$\mathbf{u} \leftarrow \mathbf{u} - \eta \left(\frac{\mathbf{m}_1}{1 - \beta_1^k} \right) \Big/ \left(\sqrt{\frac{\mathbf{m}_2}{1 - \beta_2^k}} + \varepsilon \right)$$

UniformAdam optimizer (their)

$$\mathbf{g} \leftarrow (\mathbf{I} + \lambda \mathbf{L})^{-p} \frac{\partial \Phi}{\partial \mathbf{x}},$$
$$\mathbf{m}_1 \leftarrow \beta_1 \mathbf{m}_1 + (1 - \beta_1)\mathbf{g}$$
$$\mathbf{m}_2 \leftarrow \beta_2 \mathbf{m}_2 + (1 - \beta_2)\mathbf{g}^2$$
$$\mathbf{u} \leftarrow \mathbf{u} - \frac{\eta}{(1 - \beta_1^k)\sqrt{\frac{\|\mathbf{m}_2\|_\infty}{1-\beta_2^k}}} \mathbf{m}_1$$

default values for Adam's betas in PyTorch/dr.JIT: (0.9, 0.999)

Optimization scheme



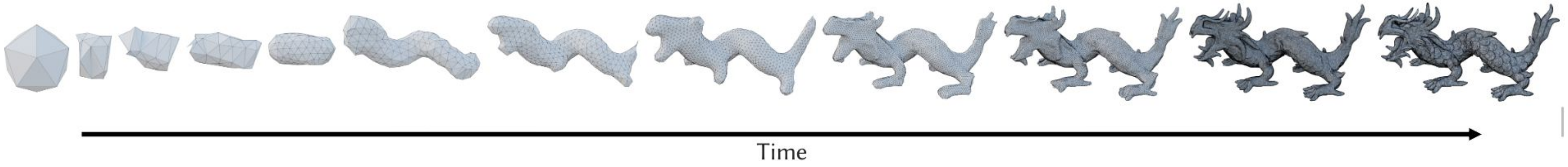
Optimization scheme

$$
\begin{aligned}
\mathbf{g} &\leftarrow (\mathbf{I} + \lambda \mathbf{L})^{-p} \frac{\partial \Phi}{\partial \mathbf{x}}, \\
\mathbf{m}_1 &\leftarrow \beta_1 \mathbf{m}_1 + (1 - \beta_1)\mathbf{g} \\
\mathbf{m}_2 &\leftarrow \beta_2\, \mathbf{m}_2 + (1 - \beta_2)\, \mathbf{g}^2 \\
\mathbf{u} &\leftarrow \mathbf{u} - \frac{\eta}{(1 - \beta_1^k)\sqrt{\frac{\|\mathbf{m}_2\|_\infty}{1-\beta_2^k}}}\, \mathbf{m}_1
\end{aligned}
$$

Computed with a **Cholesky factorization**

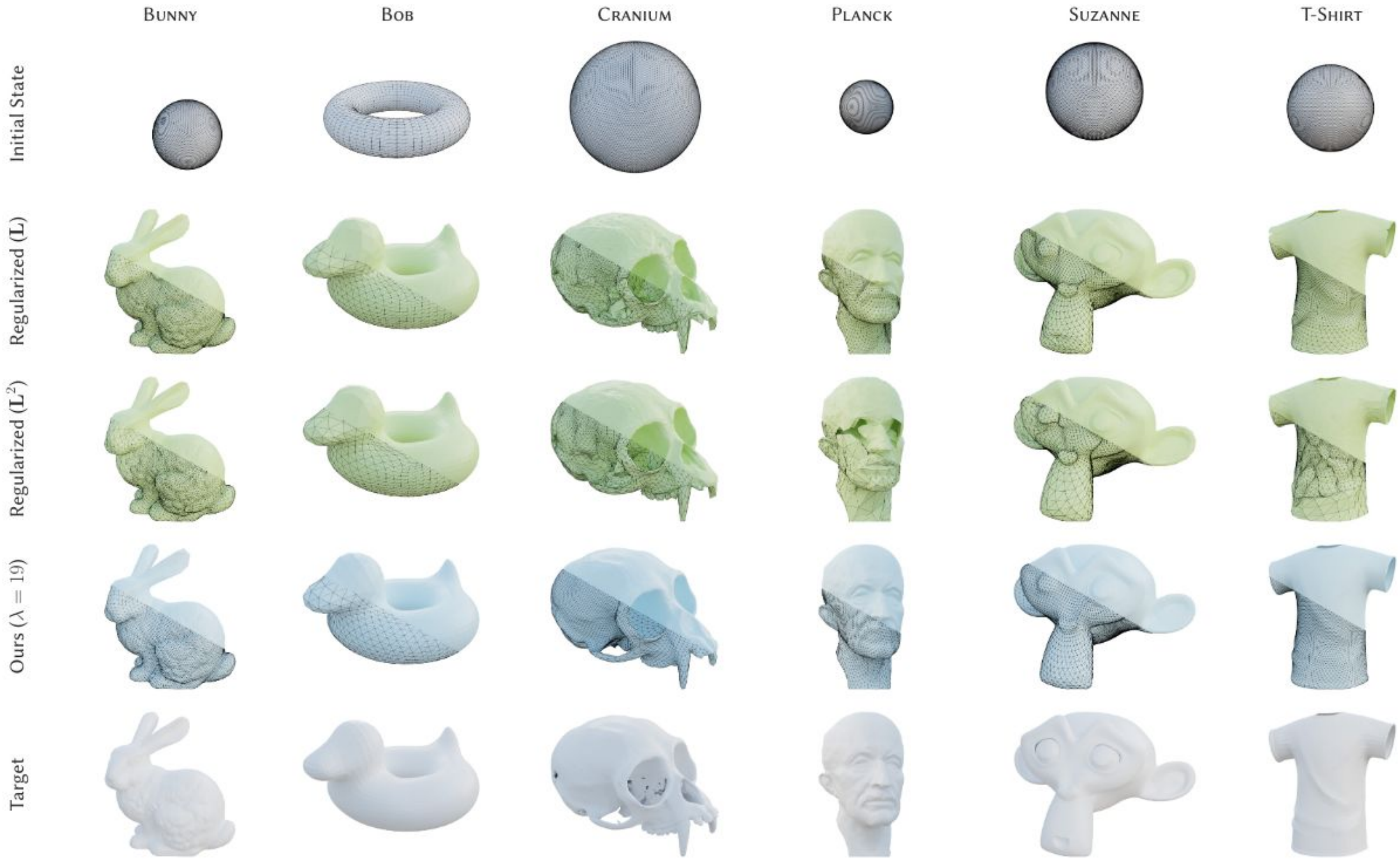
The combinatorial Laplacian is used: it **only depends on the connectivity**, not on the vertex positions. So the factorization can be **re-used across iterations**.

They also implement a remeshing strategy: at (manually-specified) iterations, they refine their mesh, using isotropic remeshing.

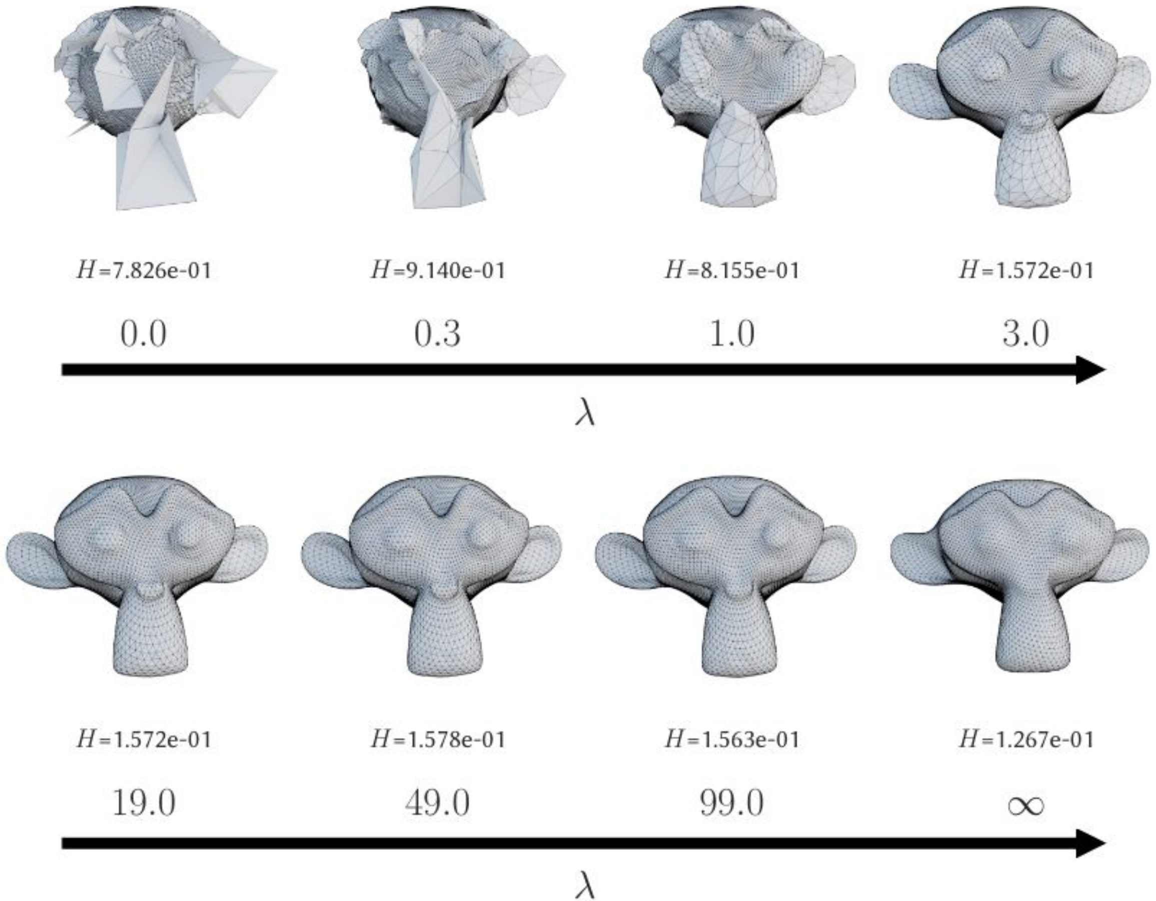


Video

Some pictures



Some pictures



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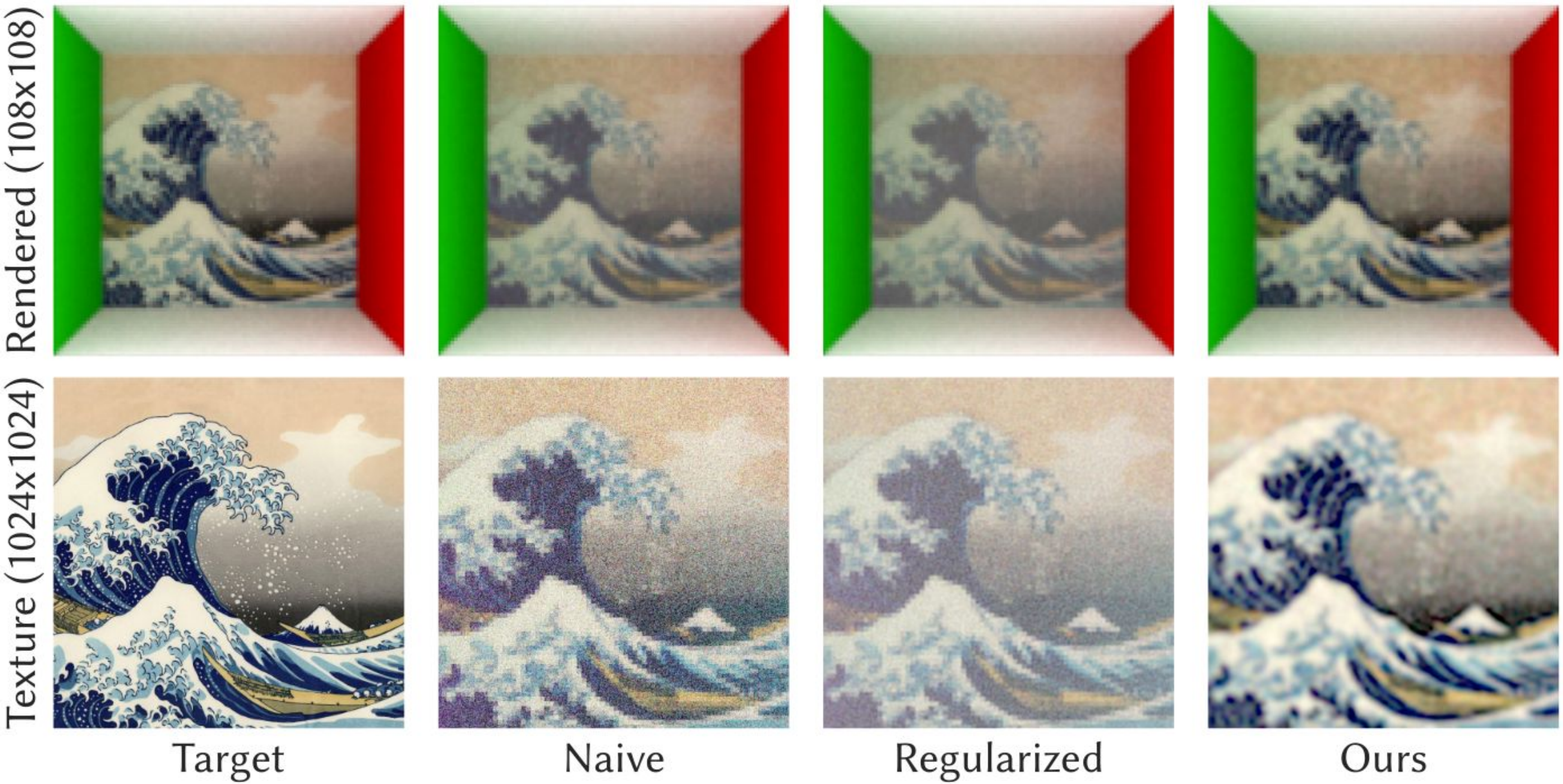

Merci !