A SIMPLE AND EFFICIENT ALGORITHM FOR DOT PATTERNS RECONSTRUCTION

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ABSTRACT

The problem of reconstructing the shape of dot patterns (sampled planar connected regions) is extensively studied in the literature. Up till now, the existing works do not provide guarantee for the correctness of the obtained solution, usually the results was validated empirically according to human perception. In this article, we present a new algorithm that guarantees reconstruction of the shape for a set of points satisfying some density conditions. Many experimental results show that the algorithm usually gives an adequate reconstruction for non-uniformly and weakly-sampled patterns. An advantage of the algorithm is its simplicity. Once the Delaunay triangulation of the input data is computed, simple rules are applied to the Delaunay edges in order to select those belonging to the reconstruction graph.

***Index Terms*—** Pattern recognition ; dot patterns ; features extraction ; shape reconstruction ; Delaunay triangulation.

1. INTRODUCTION

Reconstructing a shape of dot patterns is often required in many domains where it is needed to extract information, to detect boundaries and to build descriptors of discrete data. Without being exhaustive, shape reconstruction finds application in pattern recognition and image processing[1, 2, 3], molecular biology [4], extraction of topographical features [5], routing in wireless networks [6, 7], geographical information system [8, 9], astronomy [10].

The problem of shape reconstruction has attracted much attention in the literature during the last twenty years and progress has been done in many directions. The earliest goal of many researchers is to offer a concrete and mathematical definition of the shape notions such as diameter and convex hull. The α-shape [11] of a points set is one of the first attempt to define the shape of a set, it is one parameterized family of sub-graphs derived from Delaunay triangulation of the points set. The parameter α controls the level of detail obtained by each α-shape. The algorithm is simple but it reconstructs the correct shape only for uniformly distributed points [12]. The $\mathcal{A}$-shape concept [13] is another formalization of the shape notion, which escapes the limits of the α-shape in sense where it captures the correct shape of non-uniformly distributed data. The edges of $\mathcal{A}$-shape are selected from the Delaunay triangulation of $S \cup \mathcal{A}$, where $\mathcal{A}$ is a finite set of points which controls the level of detail captured by $\mathcal{A}$-shape. The $\mathcal{A}$-shape algorithm is designed through a double computation of the Voronoi diagram: the Voronoi diagram of the input set of points S, and then the Voronoi diagram of the set $S \cup \mathcal{A}$ from which the graph $\mathcal{A}$-shape of S is extracted, this twice computation makes it time-consuming. We also mention the algorithm of Duckham et al. [14] capturing the shape as a simple polygon (holes cannot be generated). Their algorithm removes the longest exterior edge from the Delaunay triangulation such that the external boundary of the resulting sub-triangulation forms a simple polygon.

Recently, the researchers investigate the problem of reconstructing sampled-curves from a theoretical point of view. Namely, under some sampling conditions the proposed methods guarantee the reconstruction of the curve from its samples (connecting samples according to their adjacencies on the curve). Amenta et al. [15] introduced the notion of ϵ-sample and proved that their algorithm reconstructs a ϵ-sampled curve. Subsequently improved algorithms such as [16, 17, 18] are appeared. All these algorithms provably solves the reconstruction problem for only curves but do not work when the sample points cover the whole area of a region and are not only onto boundary. Reconstruction of planar regions has been investigated in [19, 20, 21] but no guarantee has been provided for the correctness of the returned solution. We mention the work [22] which show the correctness reconstruction captured by a variational alpha-shape only for highly dense set of points.

In this article, we propose a novel simple and efficient one-step algorithm for reconstructing the shape of dot patterns. This work is motivated by three concerns: firstly, the need to capture easily the shape from the Delaunay triangulation of the data using a simple calculus. Secondly, the require to reconstruct the shape of non-uniformly and weakly sampled planar regions. Finally, the need to provide guarantees to reconstruct the correct shape under some sampling conditions. The proposed approach extracts what we call $\mathcal{A}_\delta$-Reconstruction Graph ($\mathcal{A}_\delta$-RG) from the Delaunay triangulation of the data. The main steps of the algorithm are: after the calculus of the Voronoi diagram of the input samples, the set $\mathcal{A}_\delta$, $0 \leq \delta \leq 1$, which aims to control the level of details reflected by the shape, is computed as a sub-set of the Voronoi vertices. Using simple rules, the edges of $\mathcal{A}_\delta$-RG are then extracted from the Delaunay triangulation of the input samples.

The experimental results shows the efficiency of this method to capture shapes of planar regions from non-uniformly distributed samples. From the theoretical point of view, we prove its correctness. Under the ϵ-sampling condition on the boundary of the regions to be reconstructed and some additional sampling conditions defining the distribution of the samples belonging to the interior of the regions, we show that $\mathcal{A}_\delta$-RG reconstructs the regions for $\epsilon < 1/5$ and $1/2 < \delta < 1$. For reason of space the proof of correctness is not presented here (refer to [23]).

The rest of this article is organized as follows: Section 2 defines the reconstruction problem, introduces basic notions and notations required in the rest of the paper. Section 3 deals with the concept $\mathcal{A}_\delta$-RG and the algorithm to construct it.

2. THE RECONSTRUCTION PROBLEM

Consider a closed compact region R bounded by a C^2-smooth curve ∂R, $Int(R)$ is the interior of R. Let $S = S_i \cup S_b$ be a union of two sets: S_i is a set of sample points from the interior of R ($Int(R)$), and S_b is the set of sample points from the boundary of R (∂R). We call the points of S_b the *boundary-samples* and those of S_i the *interior-samples*. In Fig. 1(b), the vertices of the polygonal curve are the boundary-sample of S_b and the other points are the interior-samples of S_i. For any two points x and y in ∂R, partition ∂R into curve segments $\Gamma(x, y)$ and $\Gamma'(x, y)$, i.e $\Gamma(x, y) \cup \Gamma'(x, y) = \partial R$ and $\Gamma(x, y) \cap \Gamma'(x, y) = \{x, y\}$, $\Gamma(x, y)$ is the curve-segment which contains the smallest number of boundary-samples.

The reconstruction graph G of the set S. The graph G is the piecewise linear consisting of all and only edges $[xy]$, $(x, y) \in S_b \times S_b$, defined as follows: $[xy]$ belongs to G iff $\Gamma(x, y) \cap S_b = \{x, y\}$, in other words the curve-segment $\Gamma(x, y)$ is empty of the boundary-samples different than x and y.

Reconstructing R from its sample set S means constructing the graph G.

Fig. 1(a) shows an example of a region R and and Fig. 1(b) illustrates the reconstruction graph G of this region from its samples.

Basic concepts and notations. We use $\mathcal{V}(S)$ and $\mathcal{D}(S)$ to denote the *Voronoi diagram* and the *Delaunay triangulation of* S, $P(x, S)$ represents the *Voronoi polygon of the sample* $x \in S$ and $CH(S)$ denotes the *convex hull of* S. The *Delaunay disc* centered at a Voronoi vertex v is denoted $D(v)$, and $B(x, y, z)$ denotes the open disc circumscribing the samples x, y and z.

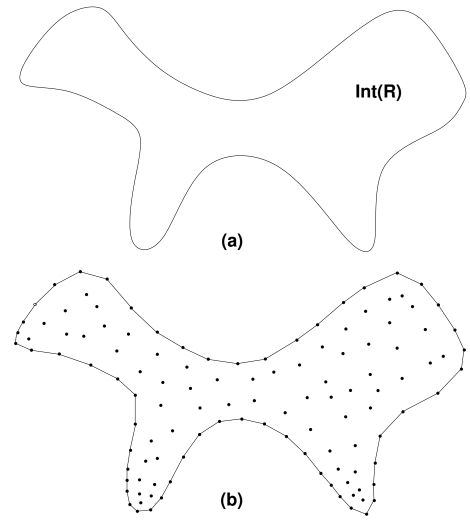

Fig. 1. (a) A region R, (b) the reconstruction graph G of R from the samples.

3. DEFINITION OF $\mathcal{A}_\delta$-RECONSTRUCTION GRAPH

3.1. Definition of the set $\mathcal{A}_\delta$

Let T be a set of the vertices of a Delaunay triangle, v be the center of the Delaunay circle circumscribing the points of T. For $0 \leq \delta \leq 1$, the set $\mathcal{A}_\delta$ is defined as follows:

Definition 1. *$\mathcal{A}_\delta$ is the subset of the vertices of $\mathcal{V}(S)$ such that a vertex v belongs to $\mathcal{A}_\delta$ if and only if there exists $x \in T$ which verify*

$$\|x - v_{min}(x)\| \leq \delta \|x - v\|$$

Where $v_{min}(x)$ denotes the nearest Voronoi vertex of $P(x, S)$ to the sample x.

The set $\mathcal{A}_\delta$ can be seen as the collection of Voronoi vertices belonging to an approximation of the external medial axis of the planar regions to be reconstructed. As shown in Fig. 2, the vertices on the approximation of the external medial axis of the sampled region are represented by empty discs. Identifying an approximation of the medial axis allows us to reconstruct the shape. The edges of the shape are those which are not destroyed by the insertion of the vertices of $\mathcal{A}_\delta$ to the Delaunay triangulation of S. To make concrete this idea in a simple way, we introduce the graph $\mathcal{A}_\delta$-RG.

3.2. Definition of the graph $\mathcal{A}_\delta$-RG

Consider two samples x and y such that $[xy]$ is a Delaunay edge of $\mathcal{D}(S)$, when $[xy]$ does not belong to the convex hull of S we use $[v_1 v_2]$ to denote the Voronoi edge separating x and y, with v_1 and v_2 are the centers of the Delaunay discs $D(v_1)$

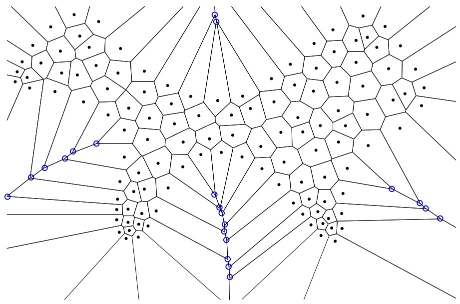

Fig. 2. The Voronoi diagram of the samples S. The Voronoi vertices represented by empty discs are on the approximation of the external medial axis.

and $D(v_2)$ circumscribing respectively $T_1 = \{x,\ y,\ z_1\}$ and $T_2 = \{x,\ y,\ z_2\}$. In the case where $[xy]$ is an edge of the convex hull of S, v_1 denotes the vertex of the Voronoi edge separating x from y, and $T_1 = \{x,\ y,\ z_1\}$ are the vertices of the Delaunay triangle associated to v_1. We define the graph $\mathcal{A}_\delta$-RG as follows.

Definition 2. *$\mathcal{A}_\delta$-RG of S is composed of the edges $[xy]$ verifying one of the conditions labeled from (1) to (5) and the edges specified by the condition (6).*

(1) $[xy] \notin CH(S)$, *$v_1 \in \mathcal{A}_\delta$ and $v_2 \in \mathcal{A}_\delta$ and $v_2 \notin B(x, y, v_1)$.*

(2) $[xy] \notin CH(S)$, *$v_1 \in \mathcal{A}_\delta$ and $v_2 \notin \mathcal{A}_\delta$ and $B(x, y, v_1) \cap S = \emptyset$.*

(3) *$[xy] \in CH(S)$ and $v_1 \in \mathcal{A}_\delta$ and $B(x, y, v_1) \cap S = \emptyset$.*

(4) *$[xy] \notin CH(S)$, $v_1 \notin \mathcal{A}_\delta$ and $v_2 \notin \mathcal{A}_\delta$, the Voronoi vertices of $\mathcal{A}_\delta$ incident to v_i, $i \in \{1,\ 2\}$, do not belong to $D(v_1) \cap D(v_2)$, and there exists a vertex $u \in \mathcal{A}_\delta$ incident to v_i such that $u \in D(v_1) \cup D(v_2)$.*

(5) *$[xy] \in CH(S)$ and $v_1 \notin \mathcal{A}_\delta$*

(6) *Consider an edge $[xy] \notin CH(S)$, with $v_1 \notin \mathcal{A}_\delta$ and $v_2 \notin \mathcal{A}_\delta$ and there exists a vertex $u \in \mathcal{A}_\delta$ incident to v_1 such that $u \in D(v_1) \cap D(v_2)$. Let $z \in S$ be the third sample on the Delaunay circle centered at v_2.*

(6.1) *$[xy]$ does not belong to $\mathcal{A}_\delta$-RG.*

(6.2) *An edge $[sz]$ with $s \in \{x, y\}$ belongs to $\mathcal{A}_\delta$-RG if and only if*

$V_{s,z} \cap \mathcal{A}_\delta = \emptyset$ and for every $w \in U_{s,z} \cap \mathcal{A}_\delta$ we have $w \notin \cup_{v \in V_{s,z}} D(v)$.

$V_{s,z}$ denotes the set of the Voronoi vertices of the edge separating s from z, $U_{s,z}$ is the set of the Voronoi vertices incident to the vertices belonging to $V_{s,z}$.

The conditions (1), (2)-(3) and (4) mean that $[xy]$ remains a Delaunay edge in respectively $\mathcal{D}(S \cup \{v_1, v_2\})$, $\mathcal{D}(S \cup \{v_1\})$ and $\mathcal{D}(S \cup \{u\})$, where u is the vertex defined in Condition (4). The condition (5) aims to detect the convex edges of the regions to be reconstructed. Condition (6) considers the case where v_1 and v_2 are closer to a vertex u incident to v_1 than x. $[xy]$ is excluded from the graph $\mathcal{A}_\delta$-RG since it is not an edge of $\mathcal{D}(S \cup \{u\})$. Instead of inserting $[xy]$ to $\mathcal{A}_\delta$-RG, the insertion of $[xz]$ and $[yz]$ to the graph is inspected. The edge, say $[xz]$ is added while $[xz]$ remains a Delaunay edge of $\mathcal{D}(S \cup \{w\})$, w being a vertex belonging to $\mathcal{A}_\delta$ and incident to a vertex of the dual Voronoi edge of $[xz]$.

In Fig. 3(a), the edge $[xy]$ satisfies the condition (1), since the dotted disc circumscribing x, y and v_1 does not contain v_2. Fig. 3(b) and Fig. 3(c) are related respectively to conditions (2) and (3). The dotted discs are empty and circumscribe v_1, x and y therefore $[xy]$ is an edge of $\mathcal{A}_\delta$-RG. Fig. 3(d) illustrates the case of the condition (4), two incident vertices u_1 and u_2 belong to $\mathcal{A}_\delta$, and neither u_1 nor u_2 belong to $D(v_1) \cap D(v_2)$. As u_1 is in $D(v_1) \cup D(v_2)$, the edge $[xy]$ verifies the condition (4). The example of Fig. 3(e) concerns the condition (6). Indeed, the vertices v_1 and v_2 are not in $\mathcal{A}_\delta$, and the vertex u_1 belongs to $\mathcal{A}_\delta$ and satisfies $u_1 \in D(v_1) \cap D(v_2)$. Thus, the edge $[xy]$ does not belong to $\mathcal{A}_\delta$-RG. We should inspect the edges $[yz]$ and $[xz]$. They both belong to the graph. For the edge $[yz]$ the reason is that the vertices of $U_{y,z}$ do not belong to $\mathcal{A}_\delta$. Concerning the edge $[xz]$ the argument is that $U_{x,z} \cap \mathcal{A}_\delta = \{u_2\}$ and $u_2 \notin D(v_2) \cup D(v_3)$.

$\mathcal{A}_\delta$-RG can be expressed as the boundary of a sub-triangulation of the Delaunay triangulation, we call it $\mathcal{A}_\delta$-complex and formally defined as follows:

Definition 3. *The $\mathcal{A}_\delta$-complex of S, denoted $\mathcal{K}_\delta(S)$, is the set of the edges belonging to $\mathcal{A}_\delta$-RG of S and the edges $[xy] \notin \mathcal{A}_\delta$-RG of S verifying $v_1 \notin \mathcal{A}_\delta$ and $v_2 \notin \mathcal{A}_\delta$ and it does not exist a vertex $u \in \mathcal{A}_\delta$ incident to v_1 or to v_2 such that $u \in D(v_1) \cap D(v_2)$.*

Observe that for $\delta_1 \leq \delta_2$, $\mathcal{K}_{\delta_1}(S) \subset \mathcal{K}_{\delta_2}(S)$. This property allows a filtration of the Delaunay triangulation, a partial ordering of the edges of the Delaunay triangulation that is useful to build topological descriptors of dot patterns, see Fig. 4 for examples.

The algorithm summarized below computes $\mathcal{A}_\delta$-RG of S, and $\mathcal{A}_\delta$-complex of S, and takes as input the set S and a parameter $0 \leq \delta \leq 1$.

Algorithm Shape Reconstruction

1. Compute the Delaunay triangulation of S.

2. Compute the Voronoi vertices (centers of the Delaunay discs) belonging to $\mathcal{A}_\delta$ using the relationship of Definition 1.

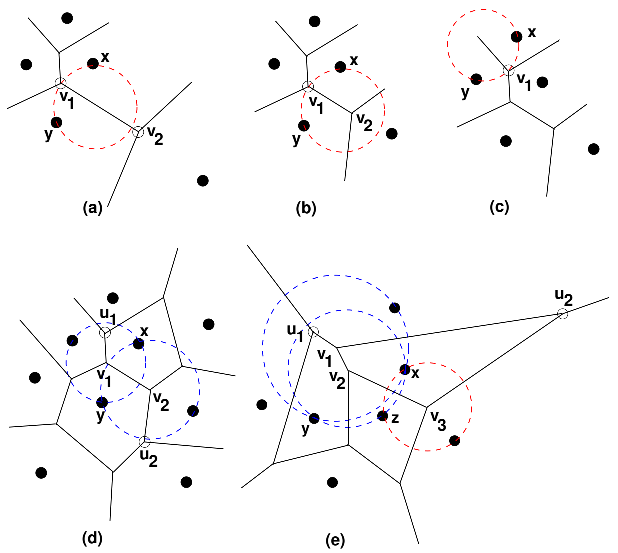

Fig. 3. Illustrations for Definition 2. (a) Condition (1), (b) condition (2), (c) condition (3), (d) condition (4), (e) condition (6).

3. Using the rules of Definition 2 and 3, extract the edges of $\mathcal{A}_\delta$-RG and the triangles and edges of $\mathcal{A}_\delta$-complex.

$\mathcal{A}_\delta$-RG is expressed in terms of a shapes family. The parameter δ makes it possible to control the level of details revealed by each shape. By varying the parameter δ from 0 to 1, $\mathcal{A}_\delta$-RG covers a finite set of shapes going from "coarse shapes" to "fine shapes". The value $\delta = 0$ corresponds to the convex hull and when δ is close to 1, all the vertices of $\mathcal{V}(S)$ belongs to $\mathcal{A}_\delta$ and thus $\mathcal{A}_\delta$-RG becomes a graph useful to reconstruct curves. The gallery of shapes shown in Fig. 4 illustrate the ability of our algorithm to reconstruct the shape of different types of non-regular and unorganized point sets. Empirical tests state that usually there is a value of δ to capture the correct shape of weakly sampled area. The range of δ which works well in practice is around 0.5, specifically $0.4 < \delta < 0.6$. From theoretical point of view, $\mathcal{A}_\delta$-RG reconstructs a planar bounded region R from its samples S under the (ϵ, δ)-sampling condition with $\frac{1}{2} < \delta$ and $\epsilon < \frac{1}{5}$. The notion of (ϵ, δ)-sampling and the proof of correctness are presented in [23].

4. CONCLUSION

We presented a new one-step algorithm for reconstructing dot patterns (sample points which cover the whole area of bounded planar regions), the returned solution is selected from a family of graphs derived from the Delaunay triangulation of the samples, we called it $\mathcal{A}_\delta$-RG with $0 \le \delta \le 1$.

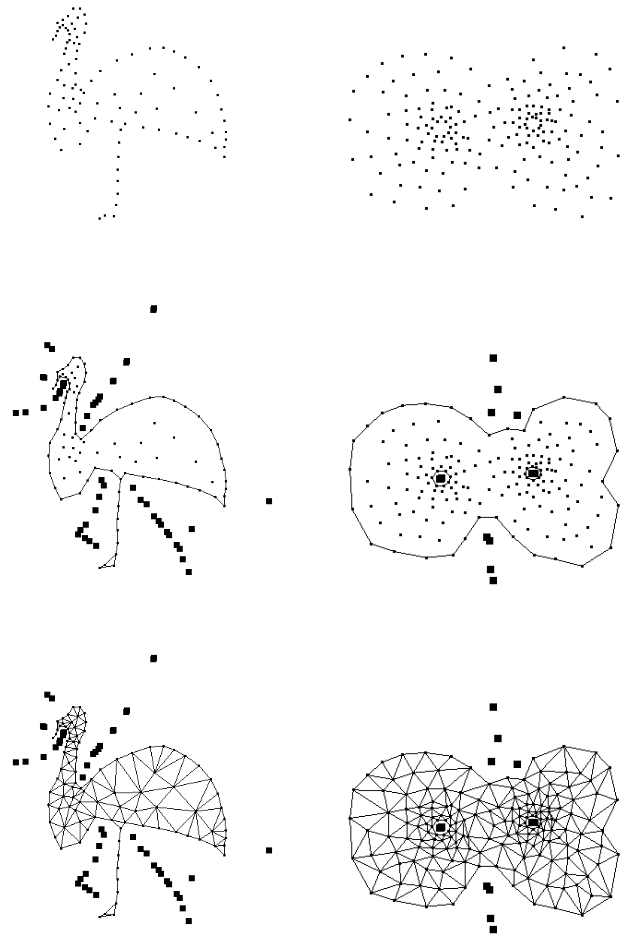

Fig. 4. $\mathcal{A}_\delta$-RG and $\mathcal{A}_\delta$-complex for two sets of points. The shapes of the left and the right column are obtained with $\delta = 0.5$.

Using simple rules, the algorithm extracts efficiently $\mathcal{A}_\delta$-RG from the Delaunay triangulation of the input samples. Its most time-consuming step is the computation of the Delaunay triangulation of the samples. From theoretical point of view, it is established that the algorithm guarantees the reconstruction under certain sampling conditions on the input samples. On the other hand, many empirical tests show that our algorithm usually provides a correct reconstruction for non-uniformly and weakly sampled planar regions.

Our future work aims to develop an effective algorithm for data in higher dimensions. All steps of the presented algorithm can be extended to higher dimensions, however, the difficulty is the computation of the higher-dimensional Delaunay triangulation. Our effort will be made to develop simple efficient algorithm avoiding the computation of the Delaunay structure.

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