Appendix: Proof of Theorems 1 and 2

In the following development, we let

$$f_{y,\lambda,p}(x) = \frac{1}{2}(x-y)^2 + \lambda|x|^p.$$

(In the maintext we simplify this denotation as $f(x)$ for the convenience of expression).

Lemma 1 *For any $y \in (\tau_p^{GST}(\lambda), +\infty)$, $f_{y,\lambda,p}(x)$ has one unique local minimum $S_p^{GST}(y_i;\lambda)$ in the range of $x \in (0,+\infty)$, which can be obtained by solving the following equation:*

$$x - y + \lambda p|x|^{p-1} = 0.$$

Proof: We prove this conclusion by contradiction.

(i) Suppose that there is no intersection point between two curves $C_1(x) = \lambda p|x|^{p-1}$ and $C_5(x) = y - x$ (see Fig. 1 for illustration), which naturally leads to

$$\lambda p|x|^{p-1} > \tau_p^{GST}(\lambda) - x$$

for any $x > 0$. This means that

$$\frac{df_{\tau_p^{GST}(\lambda),\lambda,p}(x)}{dx} = \lambda p|x|^{p-1} - (\tau_p^{GST}(\lambda) - x) > 0.$$

That is, for all $x > 0$, $f_{\tau_p^{GST}(\lambda),\lambda,p}(x)$ is monotonically increasing, which implies that

$$f_{\tau_p^{GST}(\lambda),\lambda,p}(x_p^*) > f_{\tau_p^{GST}(\lambda),\lambda,p}(0),$$

where x_p^* is defined in Eq. (24) in the maintext. This contradicts Eq. (21) in that

$$f_{\tau_p^{GST}(\lambda),\lambda,p}(x_p^*) = f_{\tau_p^{GST}(\lambda),\lambda,p}(0).$$

Suppose that there is one intersection point between $C_1(x) = \lambda|x|^{p-1}$ and $C_4(x) = y - x$. It is very easy to show that

$$\lambda p|x|^{p-1} \geqslant \tau_p^{GST}(\lambda) - x$$

for any $x > 0$, and the equality only holds in one tangent point between two curves. This also implies that $f_{\tau_p^{GST}(\lambda),\lambda,p}(x)$ is monotonically increasing in $x > 0$, and

$$f_{\tau_p^{GST}(\lambda),\lambda,p}(x_p^*) > f_{\tau_p^{GST}(\lambda),\lambda,p}(0),$$

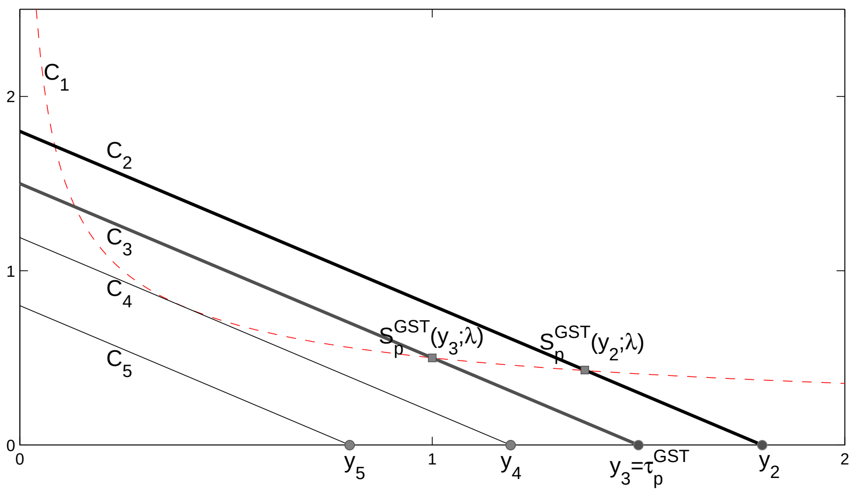

Figure 1: Illustrations for the proof of Lemma 1. $C_1(x) - C_5(x)$ denote five curves: $C_1(x) = \lambda p|x|^{p-1}, C_i(x) = y_i - x$ for $i = 2, 3, 4, 5$, where $y_3 = \tau_p^{GST}(\lambda), y_2 > y_3 > y_4 > y_5$. $C_4(x)$ is tangent with $C_1(x)$. The circles denote points $(0, y_i)$, $i = 2, 3, 4, 5$. The squares denote the intersection points $S_p^{GST}(y_i; \lambda)$, $i = 2, 3$ between $C_i(x)$ and $C_1(x)$. The square points also correspond to the minima of $f_{y_i,\lambda,p}(x)$ for $i = 2, 3$.

which also leads to contradiction with Eq. (21).

(ii) For any $y \in (\tau_p^{GST}(\lambda), +\infty)$, there are two intersection points between curves $C_1(x) = \lambda p|x|^{p-1}$ and $C_2(x) = y - x$ (See Fig. 1 for illustration).

This conclusion is evident, which can be clearly understood by Fig. 1. It can also be easily proved by contradiction. Suppose that there is one or no intersection point between two curves, and then it follows that

$$\lambda|x|^{p-1} \geqslant y - x > \tau_p^{GST}(\lambda) - x.$$

And then we have

$$f_{\tau_p^{GST}(\lambda),\lambda,p}(x_p^*) > f_{\tau_p^{GST}(\lambda),\lambda,p}(0).$$

This leads to contradiction.

(iii) For any $y \in (\tau_p^{GST}(\lambda), +\infty)$, $f_{y,\lambda,p}(x)$ has one unique local minimum $S_p^{GST}(y_i; \lambda)$ in the range of $x \in (0, +\infty)$, corresponding to the larger intersection point between $C_1(x) = \lambda p|x|^{p-1}$ and $C_3(x) = y - x$ (See Fig. 1 for illustration)..

For any $y \in (\tau_p^{GST}(\lambda), +\infty)$, denote the smaller and larger intersection points between $C_1(x)$ and $C_3(x)$ as $x = p_1$ and $x = p_2$, respectively. Based on (ii), we have that

(1) For $x \in (0, p_1)$, $\frac{df_{y,\lambda,p}(x)}{dx} = \lambda p|x|^{p-1} - (\tau_p^{GST}(\lambda) - x) > 0$, and thus $f_{y,\lambda,p}(x)$ is monotonically increasing.

(2) For $x \in (p_1, p_2)$, $\frac{df_{y,\lambda,p}(x)}{dx} < 0$, and thus $f_{y,\lambda,p}(x)$ is monotonically decreasing.

(3) For $x \in (p_2, +\infty)$, $\frac{df_{y,\lambda,p}(x)}{dx} > 0$, and thus $f_{y,\lambda,p}(x)$ is monotonically increasing.

This shows that the larger intersection point p_2 ($S_p^{GST}(y_i;\lambda)$) is the unique local minimum of $f_{y,\lambda,p}(x)$ in $x \in (0,+\infty)$, which satisfies

$$S_p^{GST}(y_i;\lambda) - y + \lambda p |S_p^{GST}(y_i;\lambda)|^{p-1} = 0.$$

The proof is completed. ■

Theorem 1 *For any* $y \in (\tau_p^{GST}(\lambda), +\infty)$*,* $f_{y,\lambda,p}(x)$ *has one unique minimum* $S_p^{GST}(y_i;\lambda)$ *in the range of* $x \in (x_p^*, +\infty)$*, which can be obtained by solving the following equation:*

$$S_p^{GST}(y_i;\lambda) - y + \lambda p |S_p^{GST}(y_i;\lambda)|^{p-1} = 0.$$

Proof: First, we prove that $x_p^* \in (p_1, +\infty)$, where p_1 is defined in (iii) in the proof of Lemma 1.

Suppose that $x_p^* \in (0, p_1)$, since $f_{y,\lambda,p}(x)$ is monotonically increasing in $(0, p_1)$, it holds that $f_{\tau_p^{GST}(\lambda),\lambda,p}(x_p^*) > f_{\tau_p^{GST}(\lambda),\lambda,p}(0)$, which leads to contradiction.

Thus $x_p^* \in (p_1, +\infty)$. Since for $x \in (p_1, p_2)$, $f_{y,\lambda,p}(x)$ is monotonically decreasing and for $x \in (p_2, +\infty)$, $f_{y,\lambda,p}(x)$ is monotonically increasing, $p_2 = S_p^{GST}(y_i;\lambda)$ is the unique global minimum of $f_{y,\lambda,p}(x)$ in $x \in (p_1, +\infty)$, and thus also in $x \in (x_p^*, +\infty)$.

The proof is completed. ■

Theorem 2 *For any* $y \in (\tau_p^{GST}(\lambda), +\infty)$*, let* $S_p^{GST}(y_i;\lambda)$ *be the unique minimum of* $f_{y,\lambda,p}(x)$ *in the range of* $(x_p^*, +\infty)$*. We have the following inequality*

$$f_{y,\lambda,p}(0) > f_{y,\lambda,p}(S_p^{GST}(y_i;\lambda)).$$

Proof: We first prove that for any $y \in (\tau_p^{GST}(\lambda), +\infty)$,

$$f_{y,\lambda,p}(0) > f_{y,\lambda,p}(x_p^*).$$

Since $y > \tau_p^{GST}(\lambda)$, we reformulate y as $\tau_p^{GST}(\lambda) + \varepsilon$, where $\varepsilon = y - \tau_p^{GST}(\lambda) > 0$. Based on Eq. (21), we know that

$$\frac{1}{2}(x_p^* - \tau_p^{GST}(\lambda))^2 + \lambda |x_p^*|^p = \frac{1}{2}(\tau_p^{GST}(\lambda))^2.$$

We then have

$$\begin{aligned}
& f_{y,\lambda,p}(x_p^*) - f_{y,\lambda,p}(0) \\
= \; & \frac{1}{2}(x_p^* - y)^2 + \lambda |x_p^*|^p - \frac{1}{2}y^2 \\
= \; & \frac{1}{2}(x_p^* - (\tau_p^{GST}(\lambda) + \varepsilon))^2 + \lambda |x_p^*|^p - \frac{1}{2}(\tau_p^{GST}(\lambda) + \varepsilon)^2 \\
= \; & \varepsilon(\tau_p^{GST}(\lambda) - x_p^*) - \varepsilon \tau_p^{GST}(\lambda) = -\varepsilon x_p^* < 0.
\end{aligned}$$

This means that

$$f_{y,\lambda,p}(0) > f_{y,\lambda,p}(x_p^*).$$

We then prove that $f_{y,\lambda,p}(0) > f_{y,\lambda,p}(S_p^{GST}(y_i;\lambda))$.

Based on the proof of Theorem 1, we know that $x_p^* \in (p_1, +\infty)$, where p_1 is defined in (iii) in the proof of Lemma 1. Since for $x \in (p_1, p_2)$, $f_{y,\lambda,p}(x)$ is monotonically decreasing and for $x \in (p_2, +\infty)$, $f_{y,\lambda,p}(x)$ is monotonically increasing, $p_2 = S_p^{GST}(y_i;\lambda)$ is the unique global minimum of $f_{y,\lambda,p}(x)$ in $x \in (p_1, +\infty)$, and thus we have

$$f_{y,\lambda,p}(S_p^{GST}(y_i;\lambda)) \leqslant f_{y,\lambda,p}(x_p^*).$$

Then it naturally follows that

$$f_{y,\lambda,p}(0) > f_{y,\lambda,p}(S_p^{GST}(y_i;\lambda)).$$

The proof is completed. ■