Supplementary Material for
Proportion Priors for Image Sequence Segmentation

Claudia Nieuwenhuis
UC Berkeley, ICSI, USA

Evgeny Strekalovskiy
TU Munich, Germany

Daniel Cremers
TU Munich, Germany

A. Convex Relaxation for the Laplace Prior.

In Section 2.2.2 of the paper, the Laplace distribution proportion prior energy

$$E_p(r_i) = \frac{\mu}{\sigma_i}\,|r_i - \bar{r}_i| = \frac{\mu}{\sigma_i}\left|\frac{a_i}{1-a_n} - \bar{r}_i\right| \tag{1}$$

is introduced and the following is stated:

Proposition 1. *The convex relaxation of* (**??**) *on the domain* $a_i, a_n \geq 0$ *and* $a_i + a_n \leq 1$ *is given by*

$$E_1(a_i, a_n) := \frac{\mu}{\sigma_i}\big|a_i - \bar{r}_i(1-a_n)\big|. \tag{2}$$

Here we give a proof of this proposition:

Proof. W.l.o.g. let $\frac{\mu}{\sigma_i} = 1$. First, E_1 is a convex lower bound on E_p since it is convex with $E_1 = E_p \cdot (1-a_n) \leq E_p$. For any other such bound $\widehat{E}_1$, by $\widehat{E}_1 \leq E_p$ it follows

$$\widehat{E}_1(0, a_n) \leq \bar{r}_i, \tag{3}$$

$$\widehat{E}_1\big(\bar{r}_i(1-a_n), a_n\big) \leq 0. \tag{4}$$

From this, $\widehat{E}_1(0,0) \leq \bar{r}_i$ and $\widehat{E}_1(0,1) \leq 0$, and therefore

$$\begin{aligned}\widehat{E}_1(0, a_n) &\leq a_n \widehat{E}_1(0,1) + (1-a_n)\widehat{E}_1(0,0) \\ &\leq \bar{r}_i(1-a_n).\end{aligned} \tag{5}$$

For $a_i \leq \bar{r}_i(1-a_n)$ we can define $\alpha := \frac{a_i}{\bar{r}_i(1-a_n)} \in [0,1]$. By convexity of $\widehat{E}_1$, and from (**??**) and (**??**) we get

$$\begin{aligned}\widehat{E}_1(a_i, a_n) &= \widehat{E}_1((1-\alpha)\cdot 0 + \alpha \cdot \bar{r}_i(1-a_n), a_n) \\ &\leq (1-\alpha)\,\widehat{E}_1(0, a_n) + \alpha\,\widehat{E}_1\big(\bar{r}_i(1-a_n), a_n\big) \\ &\leq (1-\alpha)\cdot \bar{r}_i(1-a_n) + \alpha \cdot 0 \\ &= \bar{r}_i(1-a_n) - a_i = E_1(a_i, a_n).\end{aligned}$$

Similarly, one can show $\widehat{E}_1 \leq E_1$ also for $a_i \geq \bar{r}_i(1-a_n)$. Thus, E_1 is the greatest convex lower bound on E_p. □

B. Implementation Details

In Section 2.2.2 of the paper we give the following dual formulation for the convex upper bound of the Laplace distribution prior E_2:

$$\begin{aligned}E_2 = \sup_{\alpha,\beta} \sum_{i=1}^{n-1} \Big(\alpha_i\big(a_i - \bar{r}_i(1-a_n)\big) - \beta_i(1-a_n)\Big) \\ + \frac{\varepsilon(n-1)}{1-a_n}.\end{aligned} \tag{6}$$

The duals α, β are constrained to be in the convex set

$$A := \Big\{(\alpha,\beta) \in \mathbb{R}^{2(n-1)} \;\Big|\; \beta_i \geq \tfrac{\varepsilon\sigma_i^2}{\mu^2}\alpha_i^2 \quad \forall 1 \leq i < n\Big\}. \tag{7}$$

Section 3 of the paper contains implementation details for the employed primal-dual algorithm.

B.1. Proximal Operator for a_n

For the primal-dual algorithm, in each iteration one must compute the proximal operator

$$\arg\min_{a_n} \left\{\frac{(a_n - a_n^0)^2}{2\tau} + \frac{\varepsilon(n-1)}{1-a_n}\right\}, \tag{8}$$

where $a_n^0 \in \mathbb{R}$ and $\tau > 0$ are constants. Setting the derivative w.r.t. a_n to zero, one has to solve a cubic equation. We use the method of [**?**] for this. Define $c := \tau\varepsilon(n-1)$, $v := \frac{1-a_n^0}{3}$, $w := v^3$ and $D := \frac{c}{4} + w$.

The case $D \geq 0$**.** In this case the solution is given by

$$a_n = 1 - v - z - \frac{v^2}{z} \tag{9}$$

with $z := \sqrt[3]{\frac{c}{2} + w + \sqrt{cD}} > 0$.

The case $D < 0$**.** Otherwise, the solution is

$$a_n = 1 - v + 2v\cos\Big(\tfrac{1}{3}\arccos\big(1 - \tfrac{2D}{w}\big)\Big). \tag{10}$$

B.2. Proximal Operator for α, β

The proximal operator is here

$$\begin{aligned}\arg\min_{(\alpha,\beta)\in A} \sum_{i=1}^{n-1} \frac{(\alpha-\alpha_i^0)^2}{2\tau} + \sum_{i=1}^{n-1} \frac{(\beta-\beta_i^0)^2}{2\tau} \\ + \sum_{i=1}^{n-1} \big(-\alpha_i \bar{r}_i - \beta_i \big).\end{aligned} \tag{11}$$

for some $\alpha_i^0, \beta_i^0 \in \mathbb{R}$ and $\tau > 0$ with the set A in (**??**). Define $\widehat{\alpha}_i := \alpha_i^0 + \tau \bar{r}_i$ and $\widehat{\beta}_i := \beta_i^0 + \tau$. Then the solution is given by the projection onto a parabola, separately for each i:

$$(\alpha_i, \beta_i) = \mathrm{proj}_{\beta_i \geq \widehat{\varepsilon}_i \alpha_i^2} \big(\widehat{\alpha}_i, \widehat{\beta}_i\big) \tag{12}$$

with $\widehat{\varepsilon}_i := \frac{\varepsilon \sigma_i^2}{\mu^2}$. Considering the optimality conditions for this projection leads to a cubic equation, which we again solve by the method of [**?**]:

If already $\widehat{\beta}_i \geq \widehat{\varepsilon}_i \widehat{\alpha}_i^2$, the solution is $(\alpha_i, \beta_i) = \big(\widehat{\alpha}_i, \widehat{\beta}_i\big)$. Otherwise, with $a := 2\widehat{\varepsilon}_i |\widehat{\alpha}_i|$, $b := \frac{2}{3}(1 - 2\widehat{\varepsilon}_i \widehat{\beta}_i)$ and $d := a^2 + b^3$ set

$$v := \begin{cases} c - \frac{b}{c} \quad \text{with } c = \sqrt[3]{a + \sqrt{d}} & \text{if } d \geq 0, \\ 2\sqrt{-b}\, \cos\left(\frac{1}{3} \arccos \frac{a}{\sqrt{-b}^3}\right) & \text{if } d < 0. \end{cases} \tag{13}$$

If $c = 0$ in the first case, set $v := 0$. The solution is then given by

$$\alpha_i = \left\{ \begin{matrix} \frac{v}{2\widehat{\varepsilon}_i} \frac{\widehat{\alpha}_i}{|\widehat{\alpha}_i|} & \text{if } \widehat{\alpha}_i \neq 0 \\ 0 & \text{else} \end{matrix} \right\}, \quad \beta_i = \widehat{\varepsilon}_i \alpha_i^2. \tag{14}$$

References

[1] J. P. McKelvey. Simple transcendental expressions for the roots of cubic equations. *Amer. J. Phys.*, 52(3):269–270, 1984. 1, 2

[2] T. Pock, D. Cremers, H. Bischof, and A. Chambolle. Global solutions of variational models with convex regularization. *SIAM J. Imaging Sci.*, 3:1122–1145, 2010.