Additional material for: Fast and Robust ℓ_1-averaging-based Pose Estimation for Driving Scenarios

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Abstract

We provide this additional material with the intention of helping readers to better understand the concepts presented in the manuscript. The issues addressed here are the expressions for the logarithm and the exponential map along with the analytical form of the Jacobian of the cost function that serves to generate the models.

1 On the expressions of the exponential and the logarithm maps

The link between a Lie algebra and its Lie group is called the exponential map **exp**. This map and its inverse $\mathbf{exp}^{-1} = \mathbf{log}$ allow us to move between the vector space generated by a Lie algebra and the group. It is important to remark that, for Lie groups like $\mathbb{SO}(3)$ or $\mathbb{SE}(3)$, the exponential map is surjective but not injective. In other words, all the elements of these groups can be "reached" from the algebra by the exponential map, but there are infinite elements in the algebra that will be mapped to the same group element (non-unique mapping) [1]. In the same way, **log** is a map just defined for some regions and under certain circumstances. This does not represent a problem for our method since all the models $\theta_i \in \mathbb{SE}(3)$ are closely enough to each other.

The **exp** map can be generally defined for all matrix groups $G \subset \mathbb{GL}(n,\mathbb{R})$, where $\mathbb{GL}(n,\mathbb{R})$ is the group of $n \times n$ real invertible matrices. In this way, $\mathbf{exp}(A)$ is defined as:

$$\mathbf{exp}(A) = I_n + \sum_{k\geq 1} \frac{A^k}{k!} = \sum_{k\geq 0} \frac{A^k}{k!}, \tag{1}$$

where I_n is the $n \times n$ identity matrix. It is proven that this series is absolutely convergent for any matrix [1]. However, there are explicit ways of calculating **exp** for some specific groups. This is the case of $\mathbb{SE}(3)$, the group of rigid transformations in $\mathbb{R}^3$. Here, the exponential map can be computed as:

$$\mathbf{exp}(S) = I_4 + S + \frac{(1-cos(\alpha))\,S^2}{\alpha^2} + \frac{(\alpha - sin(\alpha))\,S^3}{\alpha^3}. \tag{2}$$

This expression is an adaptation of the well known Rodrigues' formula [2]. Here, $S \in \mathfrak{se}(3)$, which is the Lie algebra corresponding to the tangent space of $\mathbb{SE}(3)$ and is parametrized as $S = \begin{bmatrix} \hat{\omega} & v \\ \vec{0} & 0 \end{bmatrix}$. Where $\omega = (\omega_x, \omega_y, \omega_z)$ is a 3-vector representing a 3D rotation and $\hat{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$ is its representation in $\mathfrak{so}(3)$, as a 3×3 real skew-symmetric matrix; $v \in \mathbb{R}^3$ is a column vector that represents the translation and $\alpha = \|\omega\|_{\ell_2}$, i.e., the amount of rotation in the direction defined by $\frac{\omega}{\alpha}$. It is also important to note that $\mathfrak{se}(3)$ is isomorphic with $\mathbb{R}^6$, which means that there exists a map that transforms between both spaces. We will take advantage of this property by representing elements of $\mathfrak{se}(3)$ as the 6-vector ψ.

In the same way, the logarithm map, for a general matrix $A \in \mathbb{GL}(n,\mathbb{R})$ has the following expression:

$$\mathbf{log}(A) = \sum_{k \geq 0} (-1)^{k+1} \frac{(A-I)^k}{k} \tag{3}$$

It is easy to see that the convergence of this series is very slow. Fortunately, in the case of the group $\mathbb{SE}(3)$ it is possible to define a close form expression for computing the **log** map, as presented below:

$$\mathbf{logSE(3)}(\theta = \begin{bmatrix} R_{3\times3} & T_{3\times1} \\ \vec{0} & 1 \end{bmatrix}) = \begin{bmatrix} r & T - \frac{\hat{r}T}{2} + \frac{2\sin(\|r\|) - \|r\|(1-\cos(\|r\|))}{2r^2\sin(\|r\|)}\hat{r}^2 T \\ \vec{0} & 1 \end{bmatrix} \tag{4}$$

where $r = \mathbf{logSO(3)}(R)$ (see Eq. 5), and $\hat{r}$ is r put as a 3×3 skew-symmetric matrix.

$$\mathbf{logSO(3)}(R) = \begin{cases} 0 & \text{if } \beta = 0 \\ \frac{\beta}{2\sin(\beta)}(R - R^T) & \text{if } \|\beta\| \in (0, \pi) \end{cases} \tag{5}$$

Here $\beta = \arccos(\frac{Trace(R)-1}{2})$.

2 The Jacobian of the cost function

Eq. 6 shows the cost function that is optimized in order to generate each of the models. To perform this optimization we apply a couple of iterations of Levenberg-Marquardt, which requires the computation of the Jacobian of 6. Usually, when a standard optimization framework is used, there is no need to provide the Jacobian of the full expression, but instead, it is enough with providing the Jacobian for $F_l^{(i)}$ and $F_r^{(i)}$ (Eq. 7).

$$C(\psi) = \sum_{i=1}^{M} \left\| \mathrm{K}\, \Pi_3 \left(\mathbf{exp}_r(\psi) \hat{X}_{l,p}^{(i)} \right) \times \hat{x}_{l,c}^{(i)} \right\|_{\ell_2}^2 + \left\| \mathrm{K} \left(\Pi_3 \left(\mathbf{exp}_r(\psi) \hat{X}_{l,p}^{(i)} \right) - \vec{B} \right) \times \hat{x}_{r,c}^{(i)} \right\|_{\ell_2}^2 = \tag{6}$$

$$\sum_{i=1}^{M} \left\| F_l^{(i)} \right\|_{\ell_2}^2 + \left\| F_r^{(i)} \right\|_{\ell_2}^2 \tag{7}$$

The Jacobian of $F_l^{(i)}$ can be computed from its derivatives as presented in Eq. 8–10. $F_r^{(i)}$ is computed in a similar fashion but accounting for the vector $\hat{B}$.

$$F_l^{(i)} = \mathrm{K}\, \Pi_3 \left(\mathbf{exp}_r(\psi) \hat{X}_{l,p}^{(i)} \right) \times \hat{x}_{l,c}^{(i)} \tag{8}$$

$$\frac{\partial F_l^{(i)}}{\partial \psi} = \partial_\psi \left[\mathrm{K} \left(\Pi_3 \mathbf{exp}_r(\psi) \hat{X}_{l,p}^{(i)} \right) \right] \times \hat{x}_{l,c}^{(i)} + \underbrace{\mathrm{K}\, \Pi_3 \left(\mathbf{exp}_r(\psi) \hat{X}_{l,p}^{(i)} \right) \times \partial_\psi \left[\hat{x}_{l,c}^{(i)} \right]}_{0} = \tag{9}$$

$$\mathrm{K}\, \Pi_3 \left(\partial_\psi \left[\mathbf{exp}_r(\psi) \right] \hat{X}_{l,p}^{(i)} \right) \times \hat{x}_{l,c}^{(i)} \tag{10}$$

Eq. 10 defines the derivative of the cost function in terms of the derivative of the exponential map. $\partial_\psi \mathbf{exp}_r(\psi)$ has to be computed according to the retraction used, in this case the Cardan map (11). This can be easily done by deriving (11) with respect to each component of ψ_i, giving rise to the six 4×4 matrices as shown by Eq. 12–17.

$$\mathbf{exp}_r(\psi) = \begin{bmatrix} \cos\psi_2 \cos\psi_3 & -\cos\psi_2 \cos\psi_3 & -\sin\psi_2 & \psi_4 \\ \cos\psi_1 \sin\psi_3 - \sin\psi_1 \sin\psi_2 \sin\psi_3 & \cos\psi_1 \cos\psi_3 + \sin\psi_1 \sin\psi_2 \sin\psi_3 & -\sin\psi_1 \cos\psi_2 & \psi_5 \\ \sin\psi_1 \sin\psi_3 + \cos\psi_1 \sin\psi_2 \cos\psi_3 & \sin\psi_1 \cos\psi_3 - \cos\psi_1 \sin\psi_2 \sin\psi_3 & \cos\psi_1 \cos\psi_2 & \psi_6 \end{bmatrix} \tag{11}$$

$$\partial_{\psi_1} \mathbf{exp}_r = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\cos\psi_1 \cos\psi_3 \sin\psi_2 - \sin\psi_1 \sin\psi_3 & -\cos\psi_3 \sin\psi_1 + \cos\psi_1 \sin\psi_2 \sin\psi_3 & -\cos\psi_1 \cos\psi_2 & 0 \\ -\cos\psi_3 \sin\psi_1 \sin\psi_2 + \cos\psi_1 \sin\psi_3 & \cos\psi_1 \cos\psi_3 + \sin\psi_1 \sin\psi_2 \sin\psi_3 & -\cos\psi_2 \sin\psi_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{12}$$

$$\partial_{\psi_2} \mathbf{exp}_r = \begin{bmatrix} -\cos\psi_3 \sin\psi_2 & \sin\psi_2 \sin\psi_3 & -\cos\psi_2 & 0 \\ -\cos\psi_2 \cos\psi_3 \sin\psi_1 & \cos\psi_2 \sin\psi_1 \sin\psi_3 & \sin\psi_1 \sin\psi_2 & 0 \\ \cos\psi_1 \cos\psi_2 \cos\psi_3 & -\cos\psi_1 \cos\psi_2 \sin\psi_3 & -\cos\psi_1 \sin\psi_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

$$\partial_{\psi_3} \mathbf{exp}_r = \begin{bmatrix} -\cos\psi_2 \sin\psi_3 & -\cos\psi_2 \cos\psi_3 & 0 & 0 \\ \cos\psi_1 \cos\psi_3 + \sin\psi_1 \sin\psi_2 \sin\psi_3 & \cos\psi_3 \sin\psi_1 \sin\psi_2 - \cos\psi_1 \sin\psi_3 & 0 & 0 \\ \cos\psi_3 \sin\psi_1 - \cos\psi_1 \sin\psi_2 \sin\psi_3 & -\cos\psi_1 \cos\psi_3 \sin\psi_2 - \sin\psi_1 \sin\psi_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

$$\partial_{\psi_4} \mathbf{exp}_r = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

$$\partial_{\psi_5} \mathbf{exp}_r = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

$$\partial_{\psi_6} \mathbf{exp}_r = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

References

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[2] V. M. Govindu. Lie-algebraic averaging for globally consistent motion estimation. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, Washington, DC, USA, 2004.