Supplementary Material: Generalised Perspective Shape from Shading with Oren-Nayar Reflectance

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In what follows, we present intermediate steps that are helpful to understand the derivation of the *normal vector* and the *four cases of the Hamilton-Jacobi equations* that correspond to our generalised perspective SfS model with Oren-Nayar reflectance.

1 Surface Parametrisation and Normal Vector

Starting our derivations in Cartesian coordinates, we can notice from Fig. 1 in the paper that

$$\overrightarrow{CX} = \overrightarrow{LX} - \overrightarrow{LC} = \begin{bmatrix} x_1 \\ x_2 \\ -(c_3 + \mathtt{f}) \end{bmatrix} - \begin{bmatrix} c_1 \\ c_2 \\ -c_3 \end{bmatrix} = \begin{bmatrix} x_1 - c_1 \\ x_2 - c_2 \\ -\mathtt{f} \end{bmatrix}. \tag{1}$$

This leads to

$$\begin{aligned} \overrightarrow{LS} &= \overrightarrow{LC} + \overrightarrow{CS} = \overrightarrow{LC} + \lambda\, \overrightarrow{CX} \\ &= \begin{bmatrix} c_1 \\ c_2 \\ -c_3 \end{bmatrix} + \lambda \begin{bmatrix} x_1 - c_1 \\ x_2 - c_2 \\ -\mathtt{f} \end{bmatrix} \\ &= \begin{bmatrix} \lambda x_1 + (1-\lambda) c_1 \\ \lambda x_2 + (1-\lambda) c_2 \\ -(c_3 + \lambda \mathtt{f}) \end{bmatrix} =: \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \end{aligned} \tag{2}$$

where $\overrightarrow{AB}$ stands for a vector notation with a starting point A and an endpoint B.

In spherical coordinates, we can describe (2) as

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} =: \mathbf{r} := r\,\mathbf{e}_r \tag{3}$$

with

$$r = \left(r_1^2 + r_2^2 + r_3^2\right)^{\frac{1}{2}}, \tag{4}$$

see Fig. 1 for details.

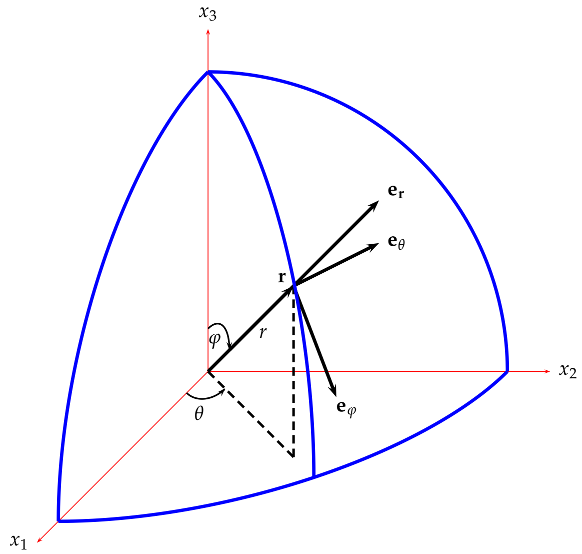

Figure 1: Illustration of parameters in spherical coordinates. $\mathbf{r} = (r_1, r_2, r_3)$ represents a vector on the sphere and r denotes the magnitude of $\mathbf{r}$ given by (4). Adapted from [1].

In order to obtain λ, we have to solve the following quadratic equation:

$$\begin{aligned} r^2 = r_1^2 + r_2^2 + r_3^2 &\overset{(2)}{=} \left[c_1 + \lambda\,(x_1 - c_1)\right]^2 + \left[c_2 + \lambda\,(x_2 - c_2)\right]^2 + (c_3 + \lambda\,\mathtt{f})^2 \\ &= \left(c_1^2 + c_2^2 + c_3^2\right) + 2\left[c_1\,(x_1 - c_1) + c_2\,(x_2 - c_2) + c_3\,\mathtt{f}\right]\lambda + \left[(x_1 - c_1)^2 + (x_2 - c_2)^2 + \mathtt{f}^2\right]\lambda^2 . \end{aligned} \tag{5}$$

Since this requires the radial depth r to be known, we switch from Cartesian coordinates to spherical coordinates and compute r from the input image directly. This can be done via the brightness equation of the Oren-Nayar model that describes the relation between the brightness values of the input image I and the normal $\mathbf{n}$ of the corresponding point on the object surface. However, before we can make use of this brightness equation, we have to express it in spherical coordinates. In particular, this requires to calculate the surface normal $\mathbf{n}$ in terms of the radial depth r. Starting from (3) and using the spherical basis described in the paper, we obtain

$$\begin{aligned}
\mathbf{n} &= \frac{\partial\,(r\mathbf{e}_r)}{\partial\theta}\times\frac{\partial\,(r\mathbf{e}_r)}{\partial\varphi}\\
&= \left(\frac{\partial r}{\partial\theta}\mathbf{e}_r + r\frac{\partial\mathbf{e}_r}{\partial\theta}\right)\times\left(\frac{\partial r}{\partial\varphi}\mathbf{e}_r + r\frac{\partial\mathbf{e}_r}{\partial\varphi}\right)\\
&= \left(\frac{\partial r}{\partial\theta}\mathbf{e}_r\times\frac{\partial r}{\partial\varphi}\mathbf{e}_r\right)+\left(\frac{\partial r}{\partial\theta}\mathbf{e}_r\times r\frac{\partial\mathbf{e}_r}{\partial\varphi}\right)+\left(r\frac{\partial\mathbf{e}_r}{\partial\theta}\times\frac{\partial r}{\partial\varphi}\mathbf{e}_r\right)+\left(r\frac{\partial\mathbf{e}_r}{\partial\theta}\times r\frac{\partial\mathbf{e}_r}{\partial\varphi}\right)\\
&= \frac{\partial r}{\partial\theta}\frac{\partial r}{\partial\varphi}\underbrace{(\mathbf{e}_r\times\mathbf{e}_r)}_{=0}+r\frac{\partial r}{\partial\theta}\left(\mathbf{e}_r\times\frac{\partial\mathbf{e}_r}{\partial\varphi}\right)+r\frac{\partial r}{\partial\varphi}\left(\frac{\partial\mathbf{e}_r}{\partial\theta}\times\mathbf{e}_r\right)+r^2\left(\frac{\partial\mathbf{e}_r}{\partial\theta}\times\frac{\partial\mathbf{e}_r}{\partial\varphi}\right)\\
&= r\frac{\partial r}{\partial\theta}\left(\mathbf{e}_r\times\frac{\partial\mathbf{e}_r}{\partial\varphi}\right)+r\frac{\partial r}{\partial\varphi}\left(\frac{\partial\mathbf{e}_r}{\partial\theta}\times\mathbf{e}_r\right)+r^2\left(\frac{\partial\mathbf{e}_r}{\partial\theta}\times\frac{\partial\mathbf{e}_r}{\partial\varphi}\right)\\
&\overset{(a)}{=} r\frac{\partial r}{\partial\theta}\left(\mathbf{e}_r\times\mathbf{e}_\varphi\right)+r\frac{\partial r}{\partial\varphi}\left(\sin\varphi\,\mathbf{e}_\theta\times\mathbf{e}_r\right)+r^2\left(\sin\varphi\,\mathbf{e}_\theta\times\mathbf{e}_\varphi\right)\\
&\overset{(b)}{=} r\frac{\partial r}{\partial\theta}\mathbf{e}_\theta + r\sin\varphi\frac{\partial r}{\partial\varphi}\mathbf{e}_\varphi - r^2\sin\varphi\,\mathbf{e}_r\,.
\end{aligned} \tag{6}$$

The intermediate step (a) in (6) can be explained by noting that

$$\frac{\partial\mathbf{e}_r}{\partial\varphi} = \begin{bmatrix}\cos\varphi\cos\theta\\ \cos\varphi\sin\theta\\ -\sin\varphi\end{bmatrix} = \mathbf{e}_\varphi \tag{7}$$

and

$$\frac{\partial\mathbf{e}_r}{\partial\theta} = \begin{bmatrix}-\sin\varphi\sin\theta\\ \sin\varphi\cos\theta\\ 0\end{bmatrix} = \sin\varphi\,\mathbf{e}_\theta\,. \tag{8}$$

In addition, we know that

$$\begin{cases}\mathbf{e}_r \times \mathbf{e}_\varphi = \mathbf{e}_\theta\\ \mathbf{e}_\varphi \times \mathbf{e}_\theta = \mathbf{e}_r\\ \mathbf{e}_\theta \times \mathbf{e}_r = \mathbf{e}_\varphi\end{cases} \tag{9}$$

since $(\mathbf{e}_\varphi, \mathbf{e}_\theta, \mathbf{e}_r)$ constitutes a right-handed coordinate system. This in turn explains step (b).

2 Oren-Nayar Brightness Equation

Starting from Section 3 of the paper, we have to calculate the trigonometric quantities that appear in the Oren-Nayar brightness equation in order to derive the Hamiltonians in spherical coordinates. Based on the Fig. 1 from the paper, we know that the light source is located at the centre of the coordinate system and thus the light direction is given by

$$\mathbf{L} = -\mathbf{e}_r\,. \tag{10}$$

On the other hand, the viewing direction at any surface point reads

$$
\begin{aligned}
\mathbf{V} &= \overrightarrow{SC} \\
&= \overrightarrow{LC} - \overrightarrow{LS} \\
&= (v_1\,\mathbf{e}_r + v_2\,\mathbf{e}_\varphi + v_3\,\mathbf{e}_\theta) - r\,\mathbf{e}_r \\
&= (v_1 - r)\,\mathbf{e}_r + v_2\,\mathbf{e}_\varphi + v_3\,\mathbf{e}_\theta\,,
\end{aligned}
\tag{11}
$$

where

$$
v_1 = \sqrt{c_1^2 + c_2^2 + c_3^2}\,, \tag{12}
$$

$$
v_2 = \arccos \frac{-c_3}{\sqrt{c_1^2 + c_2^2 + c_3^2}}\,, \tag{13}
$$

$$
v_3 = \arctan \frac{c_2}{c_1}\,. \tag{14}
$$

Knowing the surface normal $\mathbf{n}$ given by (6), the light direction $\mathbf{L}$ and the viewing direction $\mathbf{V}$ given by (10) and (11), respectively, and by making use of the relation

$$
\nabla r := \nabla_{(\theta,\varphi)} r = \frac{1}{r}\left(\frac{\partial r}{\partial \varphi}\right)\mathbf{e}_\varphi + \frac{1}{r\sin\varphi}\left(\frac{\partial r}{\partial \theta}\right)\mathbf{e}_\theta\,, \tag{15}
$$

we can reformulate all trigonometric expressions of the Oren-Nayar brightness equation as follows:

$$
\begin{aligned}
\cos(\theta_i) &= \mathbf{N}\cdot\mathbf{L} \\
&= \frac{\mathbf{n}}{|\mathbf{n}|}\cdot\mathbf{L} \\
&\overset{(6)}{\underset{(10)}{=}} \frac{\left(r\,\frac{\partial r}{\partial\theta}\,\mathbf{e}_\theta + r\sin\varphi\,\frac{\partial r}{\partial\varphi}\,\mathbf{e}_\varphi - r^2\sin\varphi\,\mathbf{e}_r\right)\cdot(-\mathbf{e}_r)}{\sqrt{r^2\left[\left(\frac{\partial r}{\partial\theta}\right)^2 + \sin^2\varphi\left(\frac{\partial r}{\partial\varphi}\right)^2 + r^2\sin^2\varphi\right]}} \\
&= \frac{r^2\sin\varphi}{r\sqrt{\left(\frac{\partial r}{\partial\theta}\right)^2 + \sin^2\varphi\left(\frac{\partial r}{\partial\varphi}\right)^2 + r^2\sin^2\varphi}} \\
&= \frac{1}{\sqrt{|\nabla r|^2 + 1}}\,,
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\cos(\theta_r) &= \mathbf{N}\cdot\mathbf{V} \\
&= \frac{\mathbf{n}}{|\mathbf{n}|}\cdot\mathbf{V} \\
&\overset{(6)}{\underset{(11)}{=}} \frac{r\frac{\partial r}{\partial\theta}v_3 + r\sin\varphi\frac{\partial r}{\partial\varphi}v_2 + (v_1 - r)(-r^2\sin\varphi)}{r^2\sin\varphi\sqrt{|\nabla r|^2+1}} \\
&= \frac{\frac{\partial r}{\partial\theta}v_3 + \sin\varphi\frac{\partial r}{\partial\varphi}v_2 + r^2\sin\varphi - v_1 r\sin\varphi}{r\sin\varphi\sqrt{|\nabla r|^2+1}} \\
&= \frac{\frac{1}{r\sin\varphi}\frac{\partial r}{\partial\theta}v_3 + \frac{1}{r}\frac{\partial r}{\partial\varphi}v_2 + r - v_1}{\sqrt{|\nabla r|^2+1}},
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\sin(\theta_i) &= \sqrt{1-(\mathbf{N}\cdot\mathbf{L})^2} \\
&\overset{(16)}{=} \sqrt{1-\left(\frac{1}{\sqrt{|\nabla r|^2+1}}\right)^2} \\
&= \sqrt{1-\frac{1}{|\nabla r|^2+1}} = \frac{|\nabla r|}{\sqrt{|\nabla r|^2+1}},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\sin(\theta_r) &= \sqrt{1-(\mathbf{N}\cdot\mathbf{V})^2} \\
&\overset{(17)}{=} \sqrt{1-\left(\frac{\frac{1}{r\sin\varphi}\frac{\partial r}{\partial\theta}v_3 + \frac{1}{r}\frac{\partial r}{\partial\varphi}v_2 + r - v_1}{\sqrt{|\nabla r|^2+1}}\right)^2} \\
&= \sqrt{1-\frac{\left(\frac{1}{r\sin\varphi}\frac{\partial r}{\partial\theta}v_3 + \frac{1}{r}\frac{\partial r}{\partial\varphi}v_2 + r - v_1\right)^2}{|\nabla r|^2+1}}.
\end{aligned}
\tag{19}
$$

In order to project the vectors **L** and **V** onto the (x_1, x_2)-plane, we have to put $\varphi = \frac{\pi}{2}$ in the orthonormal basis defined in the paper and we have to reduce v_1 and r defined in (12) and (4) to the first two components. In this way, defining these projections of **L** and **V** as $\hat{\mathbf{l}}$ and $\hat{\mathbf{v}}$, respectively, we can attain

$$\cos(\varphi_i - \varphi_r) = \hat{\mathbf{l}} \cdot \hat{\mathbf{v}} = \sqrt{r_1^2 + r_2^2} - \sqrt{c_1^2 + c_2^2}\,. \tag{20}$$

3 Hamilton-Jacobi Equations in Spherical Coordinates

In order to derive the Hamilton-Jacobi equations (HJEs) corresponding to the Oren-Nayar brightness equation, we will use the formulas of the previous section. Thus, we obtain the following four cases:

Case 1: $\theta_i \geq \theta_r$ and $(\varphi_i - \varphi_r) \in [0, \frac{\pi}{2}) \cup (\frac{3}{2}\pi, 2\pi]$

In this case, we have the following implication:

$$\max[0, \cos(\varphi_i - \varphi_r)] = \cos(\varphi_i - \varphi_r) \tag{21}$$

and hence we attain

$$\begin{aligned}
& I(\mathbf{x}) = \frac{1}{r^2}\cos(\theta_i)\left(A + B\cos(\varphi_i - \varphi_r)\sin(\theta_i)\frac{\sin(\theta_r)}{\cos(\theta_r)}\right) \\
\Leftrightarrow \quad & r^2 I = A\cos(\theta_i) + B\cos(\varphi_i - \varphi_r)\cos(\theta_i)\sin(\theta_i)\frac{\sin(\theta_r)}{\cos(\theta_r)} \\
\Leftrightarrow \quad & r^2 I = A(\mathbf{N}\cdot\mathbf{L}) + B(\hat{\mathbf{l}}\cdot\hat{\mathbf{v}})\,(\mathbf{N}\cdot\mathbf{L})\sqrt{1-(\mathbf{N}\cdot\mathbf{L})^2}\cdot\frac{\sqrt{1-(\mathbf{N}\cdot\mathbf{V})^2}}{(\mathbf{N}\cdot\mathbf{V})} \\
\Leftrightarrow \quad & r^2 I = \frac{A}{\sqrt{|\nabla r|^2+1}} + \frac{B(\hat{\mathbf{l}}\cdot\hat{\mathbf{v}})}{\sqrt{|\nabla r|^2+1}}\cdot\frac{|\nabla r|}{\sqrt{|\nabla r|^2+1}}\cdot \\
& \sqrt{1 - \frac{\left(\frac{1}{r\sin\varphi}\frac{\partial r}{\partial\theta}v_3 + \frac{1}{r}\frac{\partial r}{\partial\varphi}v_2 + r - v_1\right)^2}{|\nabla r|^2+1}}\cdot\frac{\sqrt{|\nabla r|^2+1}}{\left(\frac{1}{r\sin\varphi}\frac{\partial r}{\partial\theta}v_3 + \frac{1}{r}\frac{\partial r}{\partial\varphi}v_2 + r - v_1\right)} \\
\Leftrightarrow \quad & r^2 I - \frac{A}{\sqrt{|\nabla r|^2+1}} - \frac{B(\hat{\mathbf{l}}\cdot\hat{\mathbf{v}})|\nabla r|}{|\nabla r|^2+1}\cdot\frac{\sqrt{|\nabla r|^2+1-\left(\frac{1}{r\sin\varphi}\frac{\partial r}{\partial\theta}v_3 + \frac{1}{r}\frac{\partial r}{\partial\varphi}v_2 + r - v_1\right)^2}}{\left(\frac{1}{r\sin\varphi}\frac{\partial r}{\partial\theta}v_3 + \frac{1}{r}\frac{\partial r}{\partial\varphi}v_2 + r - v_1\right)} = 0\,.
\end{aligned} \tag{22}$$

Case 2: $\theta_i < \theta_r$ and $(\varphi_i - \varphi_r) \in [0, \frac{\pi}{2}) \cup (\frac{3}{2}\pi, 2\pi]$

Since (21) is still true, we obtain

$$
\begin{aligned}
& I(\mathbf{x}) = \frac{1}{r^2}\cos(\theta_i)\left(A + B\cos(\varphi_i - \varphi_r)\sin(\theta_r)\frac{\sin(\theta_i)}{\cos(\theta_i)}\right) \\
\Leftrightarrow \quad & r^2 I = A\cos(\theta_i) + B\cos(\varphi_i - \varphi_r)\sin(\theta_i)\sin(\theta_r) \\
\Leftrightarrow \quad & r^2 I = A(\mathbf{N}\cdot\mathbf{L}) + B(\hat{\mathbf{l}}\cdot\hat{\mathbf{v}})\sqrt{1-(\mathbf{N}\cdot\mathbf{L})^2}\sqrt{1-(\mathbf{N}\cdot\mathbf{V})^2} \\
\Leftrightarrow \quad & r^2 I - \frac{A}{\sqrt{|\nabla r|^2+1}} - \frac{B(\hat{\mathbf{l}}\cdot\hat{\mathbf{v}})|\nabla r|}{\sqrt{|\nabla r|^2+1}}\sqrt{1 - \frac{\left(\frac{1}{r\sin\varphi}\frac{\partial r}{\partial\theta}v_3 + \frac{1}{r}\frac{\partial r}{\partial\varphi}v_2 + r - v_1\right)^2}{|\nabla r|^2+1}} = 0 \\
\Leftrightarrow \quad & r^2 I - \frac{A}{\sqrt{|\nabla r|^2+1}} - \frac{B(\hat{\mathbf{l}}\cdot\hat{\mathbf{v}})|\nabla r|}{|\nabla r|^2+1}\sqrt{|\nabla r|^2 + 1 - \left(\frac{1}{r\sin\varphi}\frac{\partial r}{\partial\theta}v_3 + \frac{1}{r}\frac{\partial r}{\partial\varphi}v_2 + r - v_1\right)^2} = 0\,.
\end{aligned}
\tag{23}
$$

Case 3: $\theta_i = \theta_r$ and $\varphi_i = \varphi_r$

In this particular case, we have a different implication: by defining $\theta := \theta_i = \theta_r = \alpha = \beta$, we have

$$
\varphi_i = \varphi_r \Rightarrow \max[0, \cos(\varphi_i - \varphi_r)] = \max[0, \cos(0)] = 1\,. \tag{24}
$$

So, we obtain the HJE

$$
\begin{aligned}
& I(\mathbf{x}) = \frac{1}{r^2}\cos(\theta)\left(A + B\frac{\sin(\theta^2)}{\cos(\theta)}\right) \\
\Leftrightarrow \quad & r^2 I = A\cos(\theta) + B\sin(\theta^2) \\
\Leftrightarrow \quad & r^2 I = \frac{A}{\sqrt{|\nabla r|^2+1}} + \frac{B|\nabla r|^2}{|\nabla r|^2+1} \\
\Leftrightarrow \quad & r^2 I(|\nabla r|^2+1) - A\sqrt{|\nabla r|^2+1} - B|\nabla r|^2 = 0\,.
\end{aligned}
\tag{25}
$$

Case 4: For any θ_i, θ_r, and $(\varphi_i - \varphi_r) \in [\frac{\pi}{2}, \frac{3}{2}\pi]$

In this case, we have the trivial implication:

$$
\max[0, \cos(\varphi_i - \varphi_r)] = 0\,. \tag{26}
$$

Because of (26), we end up with the following HJE:

$$\begin{aligned} & I(\mathbf{x}) = \frac{1}{r^2} A \cos\theta_i \\ \Leftrightarrow \quad & r^2 I = A \cos\theta_i \\ \Leftrightarrow \quad & r^2 I = A(\mathbf{N} \cdot \mathbf{L}) \\ \Leftrightarrow \quad & r^2 I = \frac{A}{\sqrt{|\nabla r|^2 + 1}} \\ \Leftrightarrow \quad & r^2 I \sqrt{|\nabla r|^2 + 1} - A = 0 \\ \Leftrightarrow \quad & I \sqrt{|\nabla r|^2 + 1} - \frac{A}{r^2} = 0\,. \end{aligned} \tag{27}$$

References

[1] S. Galliani, Y.-C. Ju, M. Breuß, and A. Bruhn. Generalised perspective shape from shading in spherical coordinates. In *Proc. Scale Space and Variational Methods in Computer Vision (SSVM)*, pages 222–233, 2013.