Normal and misère play of multiplayer games with preference

Games and Graphs Workshop

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University Lyon~1

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Background

- Early studies
  - Normal and misère NIM
  - Multiplayer game with preference
    - Includes Li's theory

Nimber

(3, 2, 4)

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010

100

101

- Calculate mod-2 sum of the number of stones of each heap in binary notation without carry

$3 \oplus 2 \oplus 4 = 5$

Normal NIM

P-position of normal NIM:

$$n_1 \oplus n_2 \oplus \ldots \oplus n_k = 0$$

Misère NIM

P-position of misère NIM:

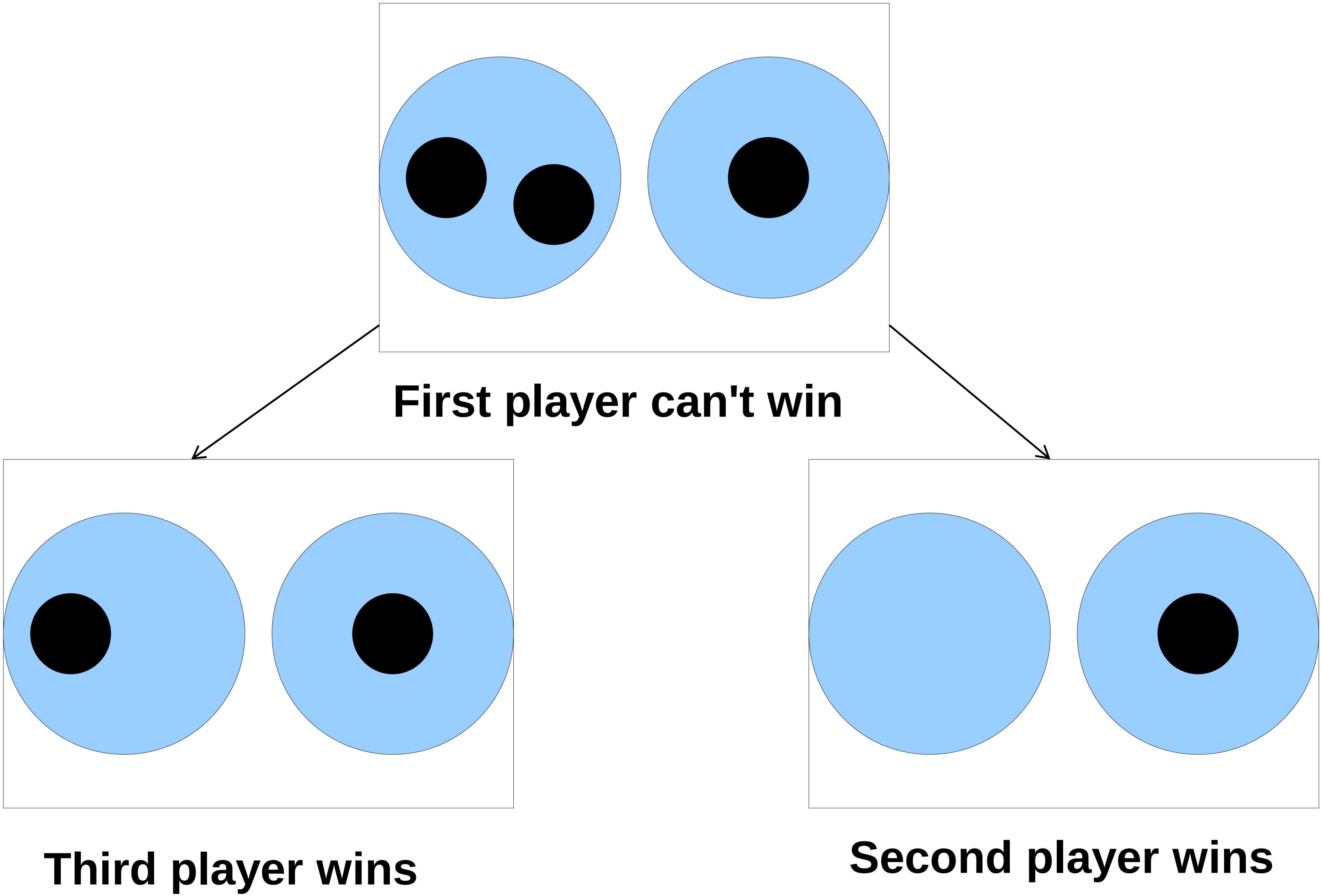
$$\begin{cases} n_1 \oplus n_2 \oplus \ldots \oplus n_k = 0 (\exists n_i > 1) \\ n_1 \oplus n_2 \oplus \ldots \oplus n_k = 1 (\forall n_i \leq 1) \end{cases}$$

Background

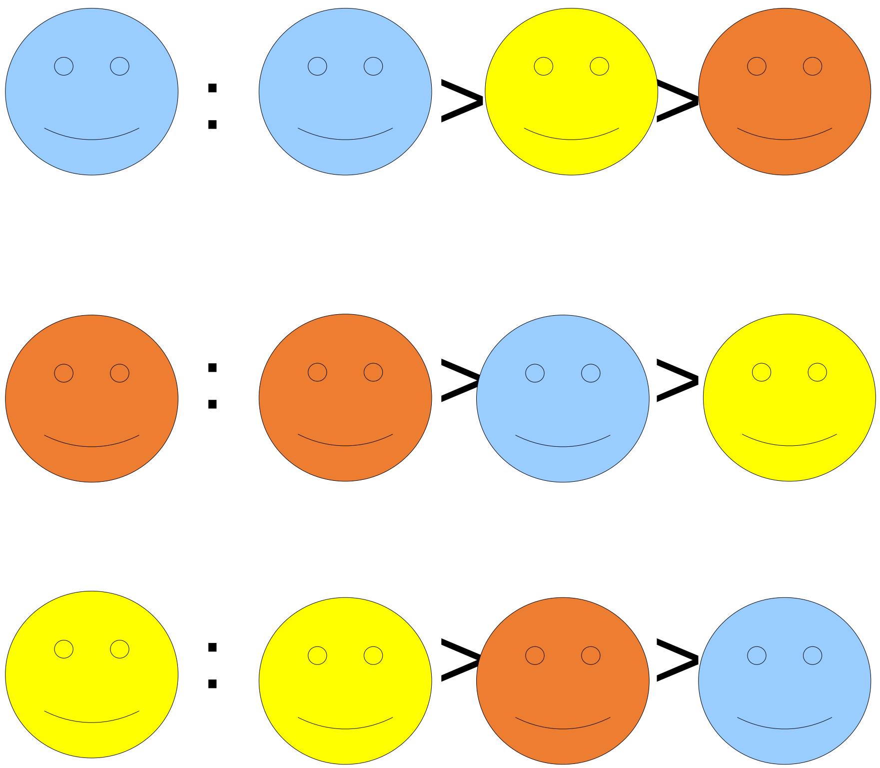
- Early studies
  - Normal and misère NIM

  - Multiplayer game with preference
    - Includes Li's theory

3-player NIM



Preference



Each player has a total "preference" ordering.

If player X has preference order $A > B$ then it is better for X that player A moves last than player B moves last.

※Assuming players behave optimally for her "preference".

Definitions

$N(A)$: Next player of player A

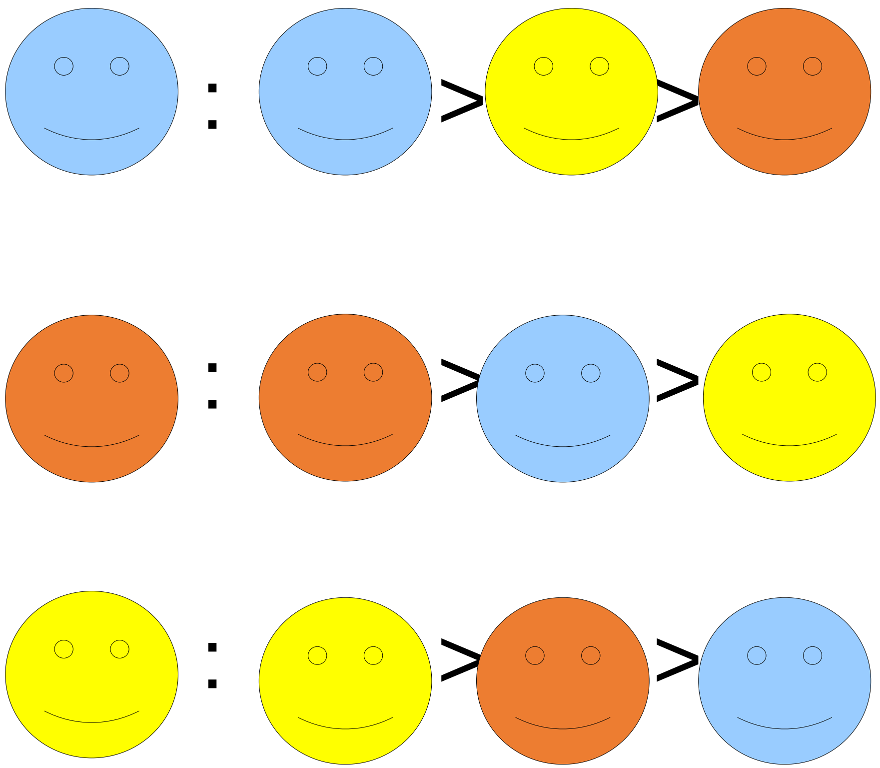
$N^{-1}(A)$: Previous player of player A

$$N^2(A) = N\big(N(A)\big), N^3(A) = N\big(N^2(A)\big), \ldots$$

$$N^{-2}(A) = N^{-1}\big(N^{-1}(A)\big), N^{-3}(A) = N^{-1}\big(N^{-2}(A)\big), \ldots$$

Note that $N^0(A) = N^n(A) = A$.

Preference



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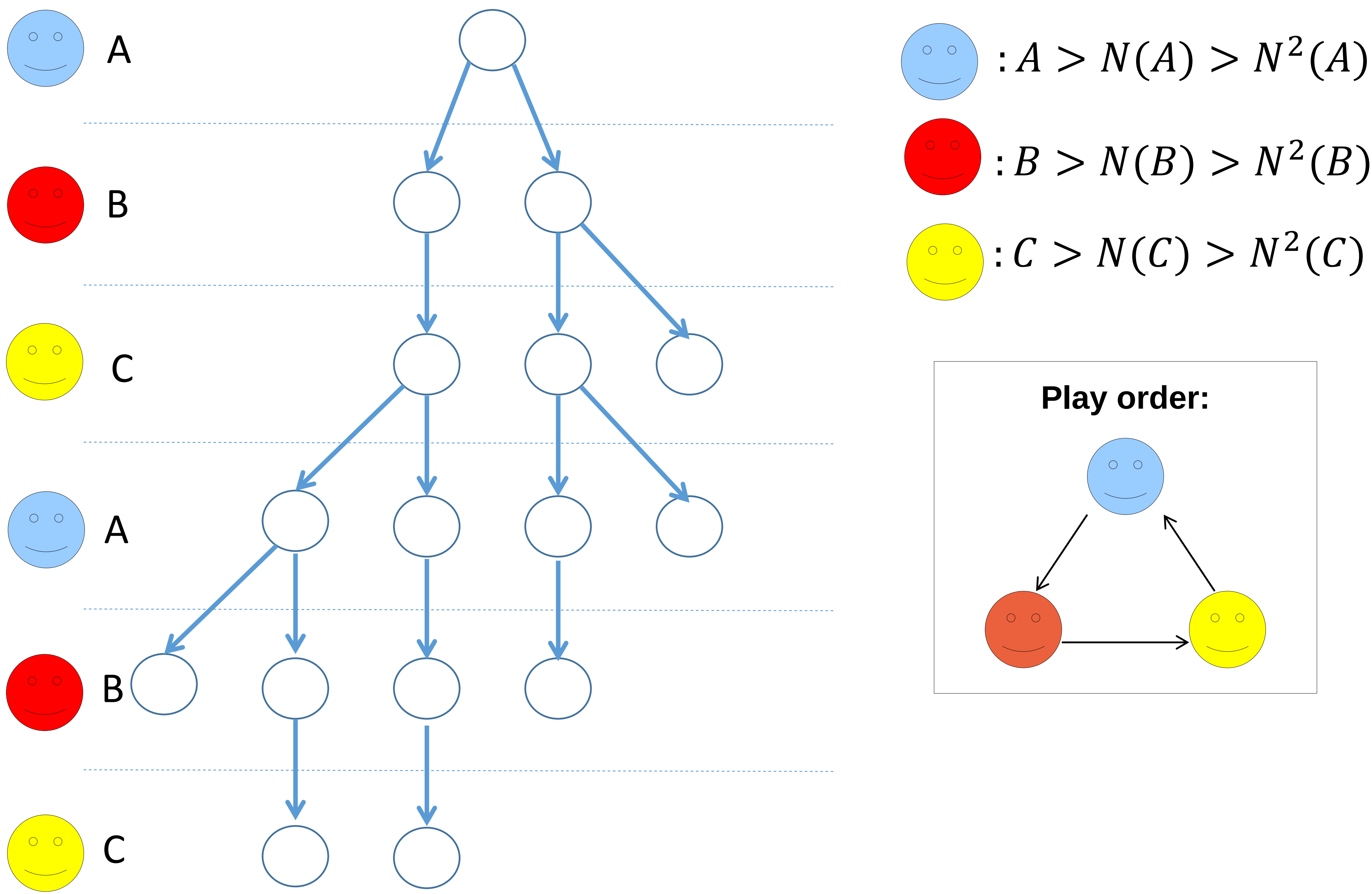
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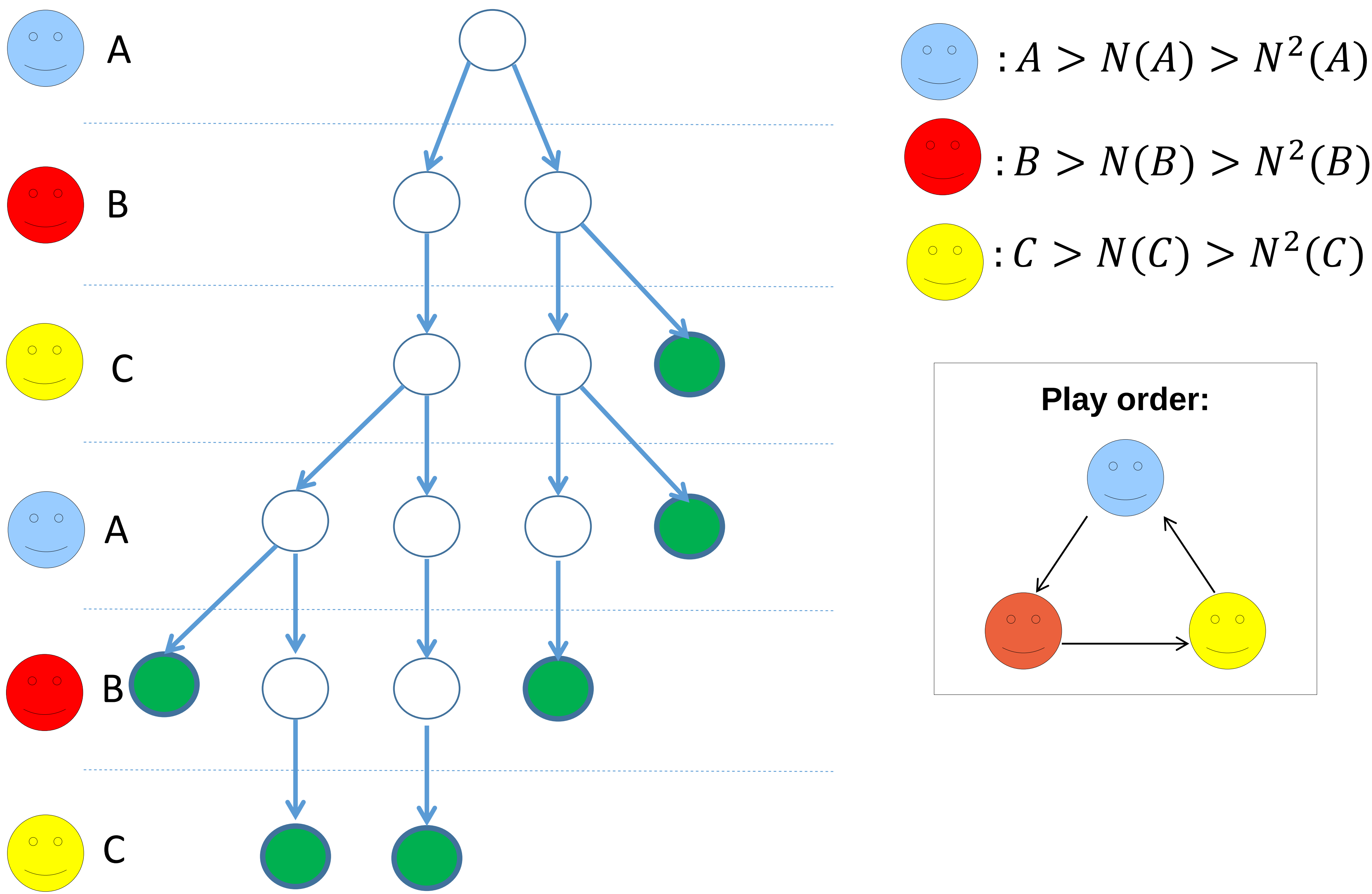
: $A > N(A) > N^2(A)$
: $B > N(B) > N^2(B)$
: $C > N(C) > N^2(C)$
Play order: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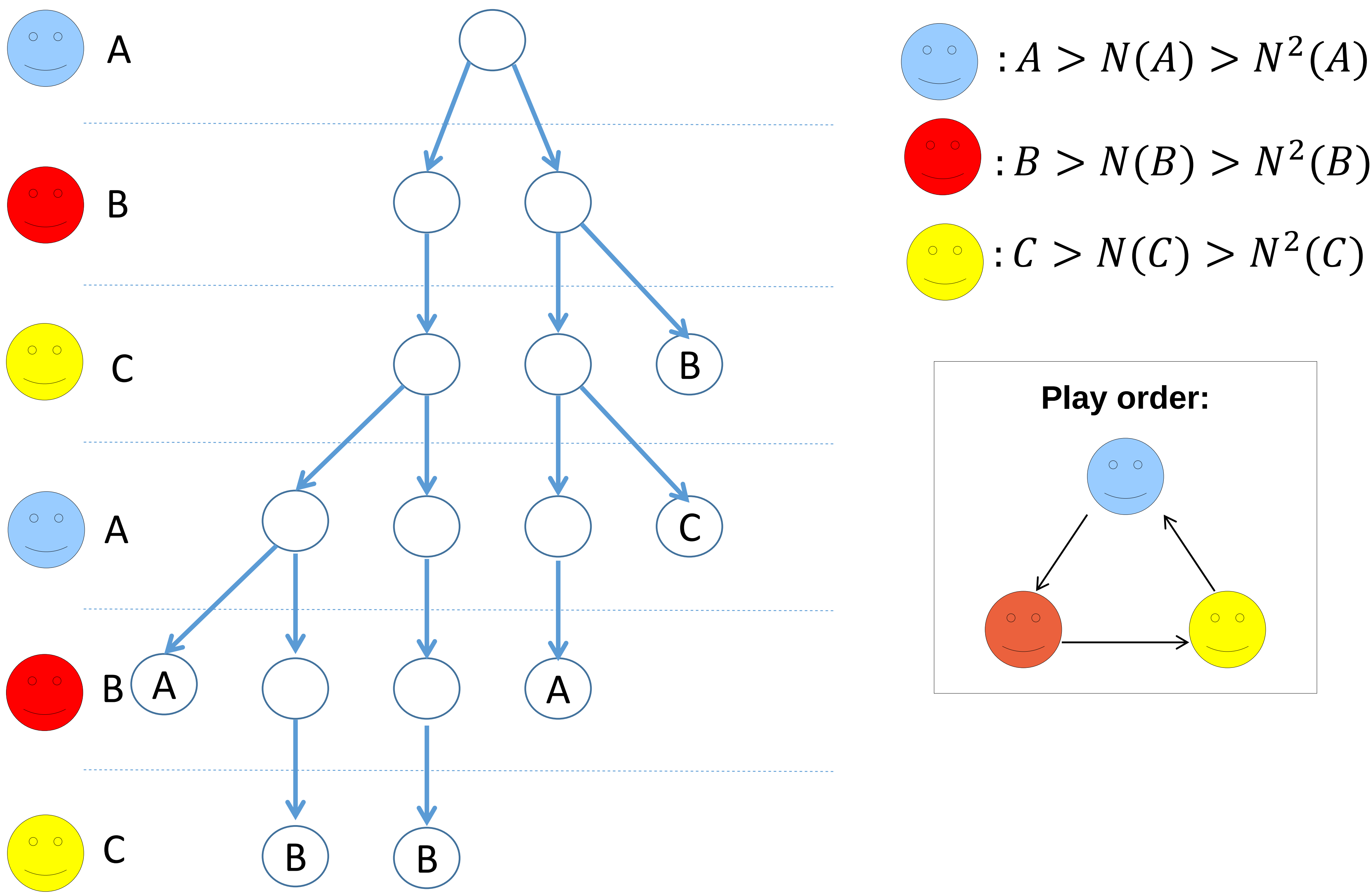

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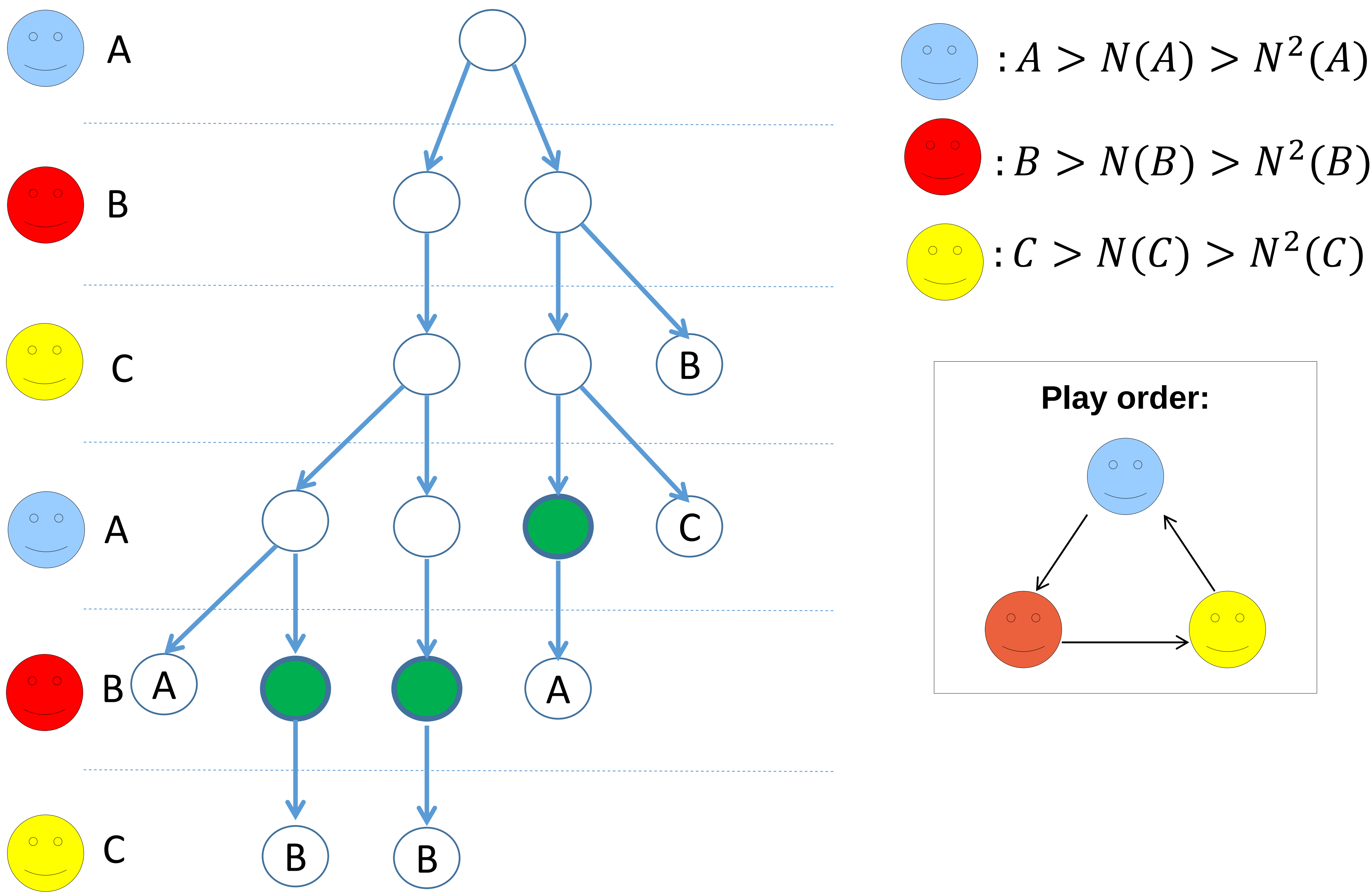
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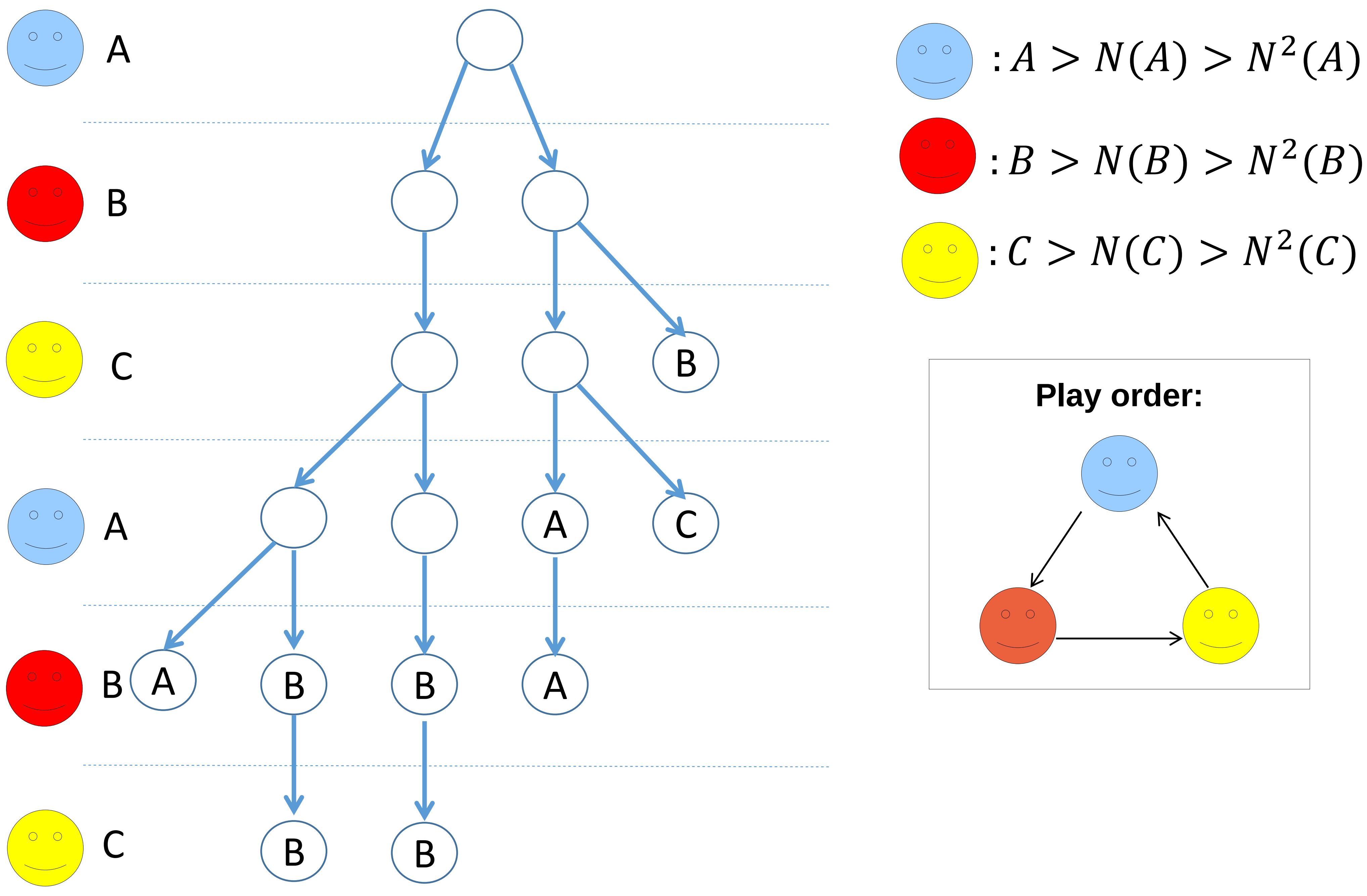

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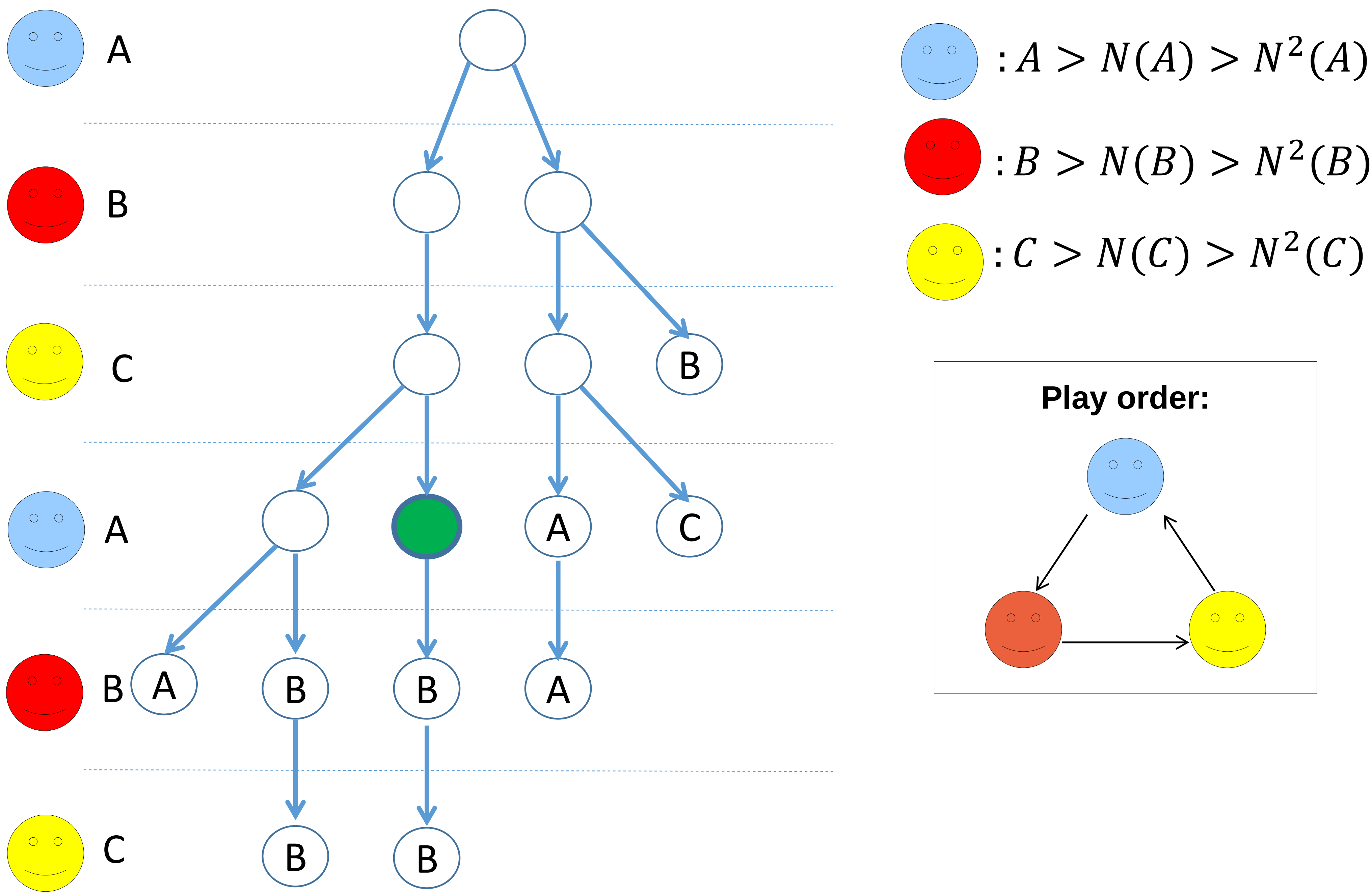
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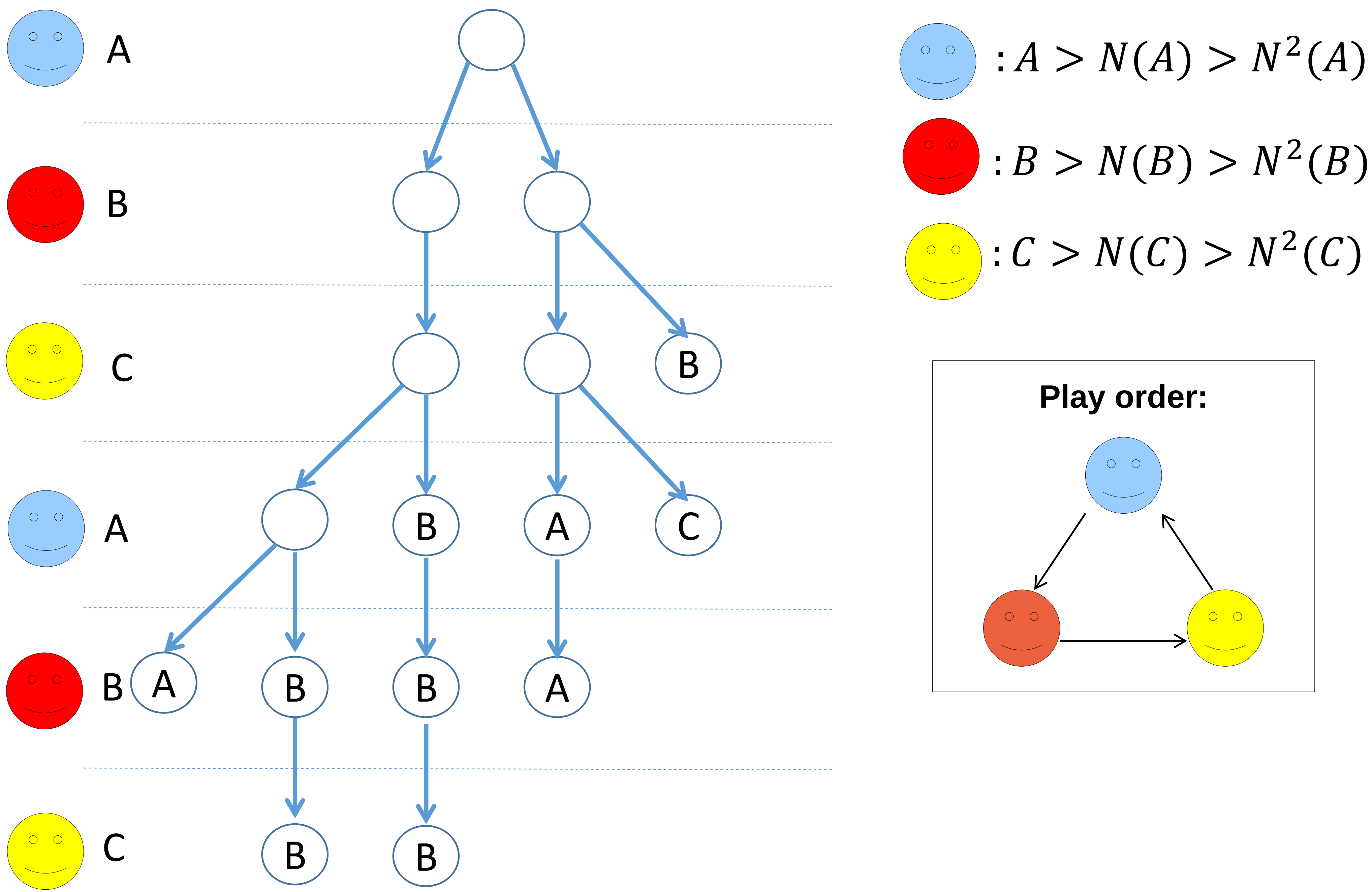

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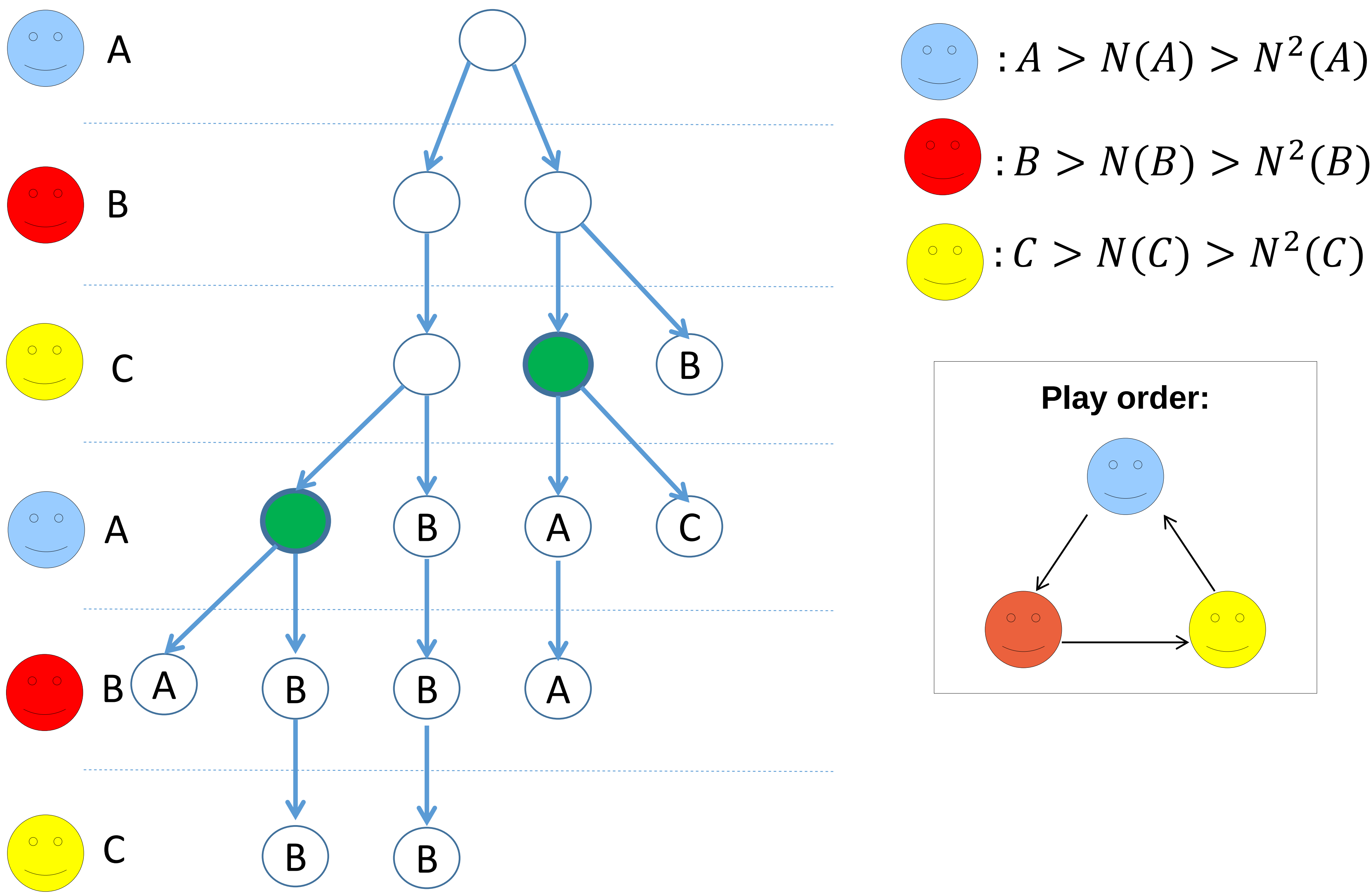

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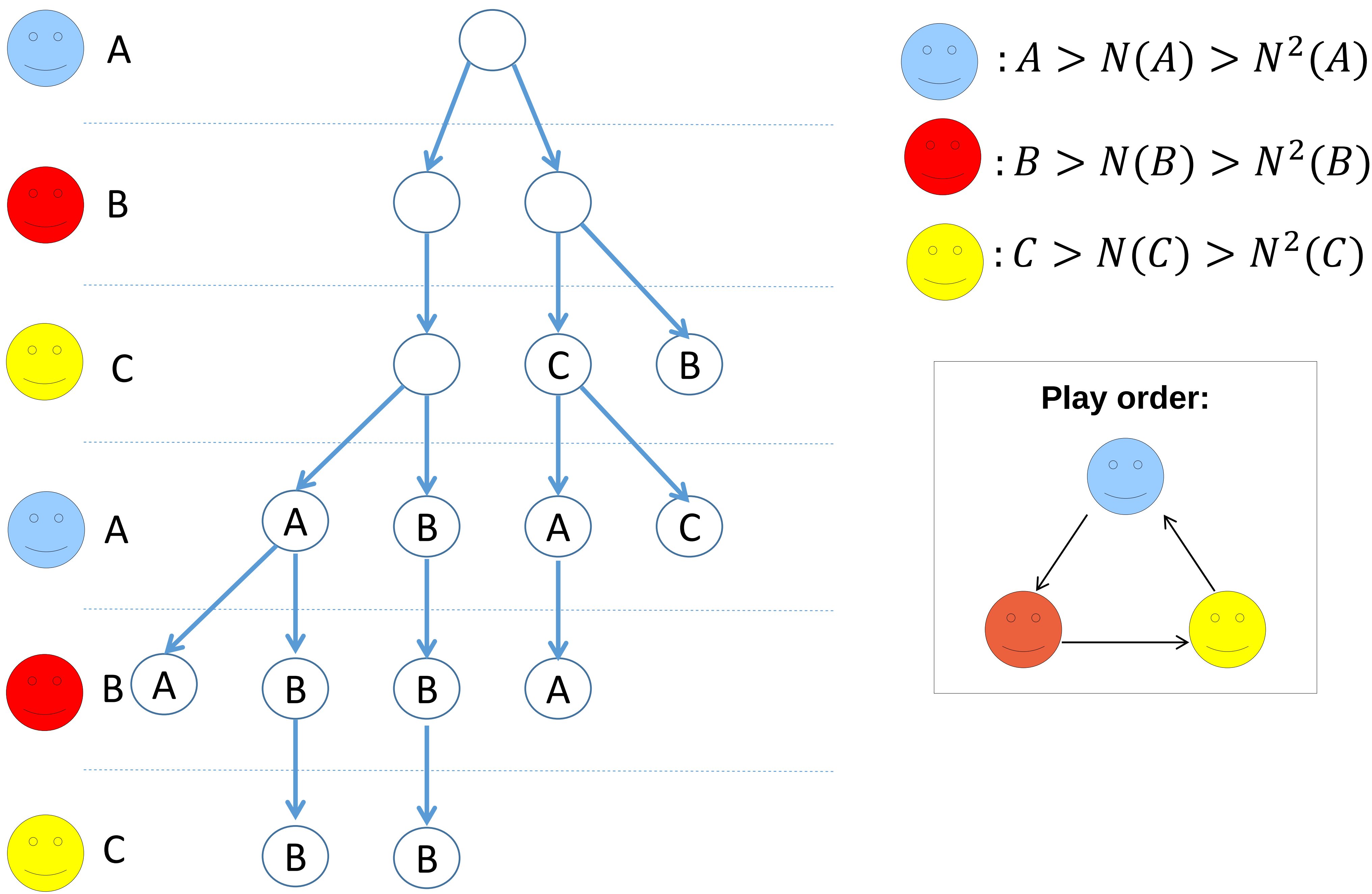
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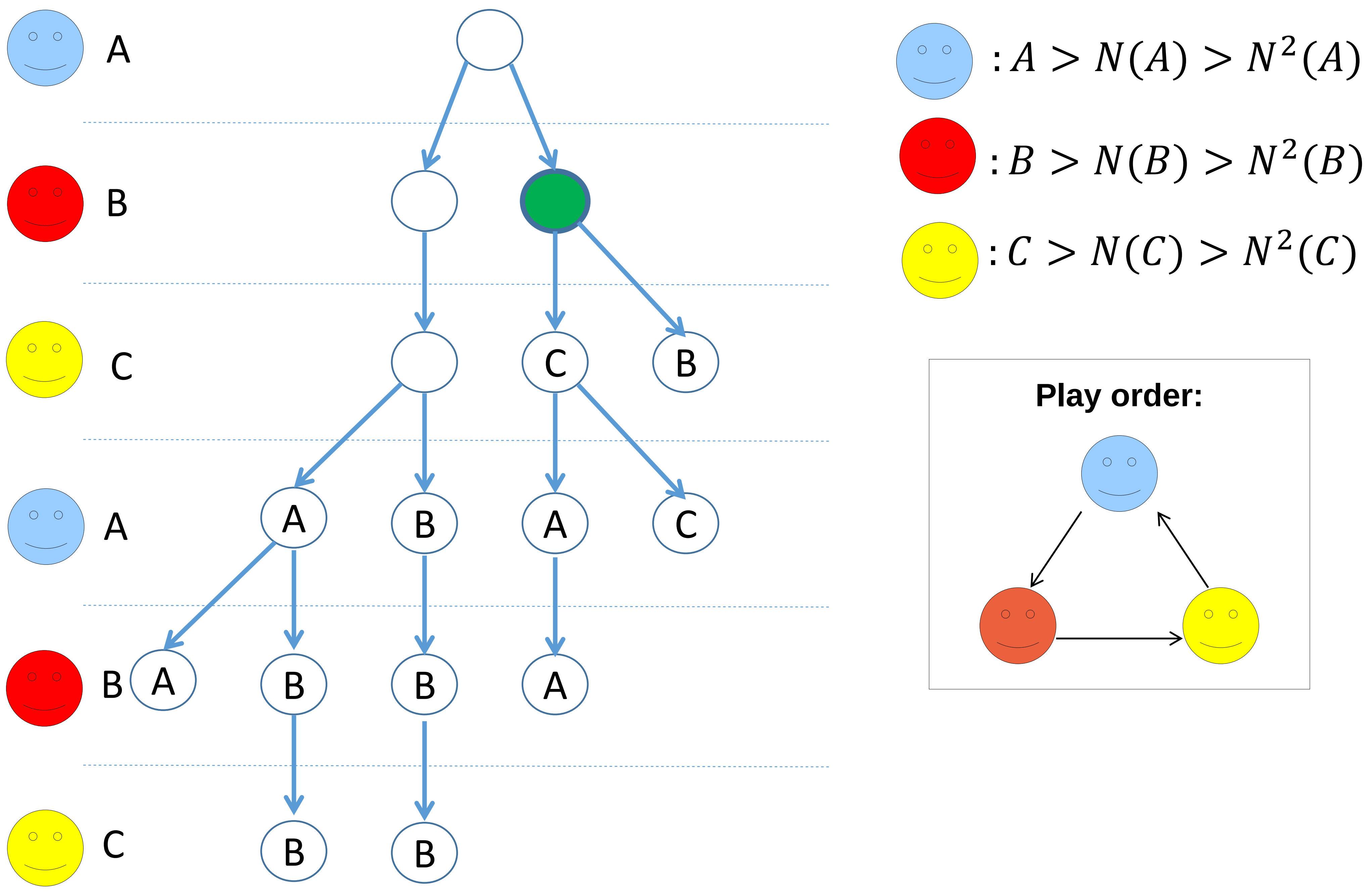
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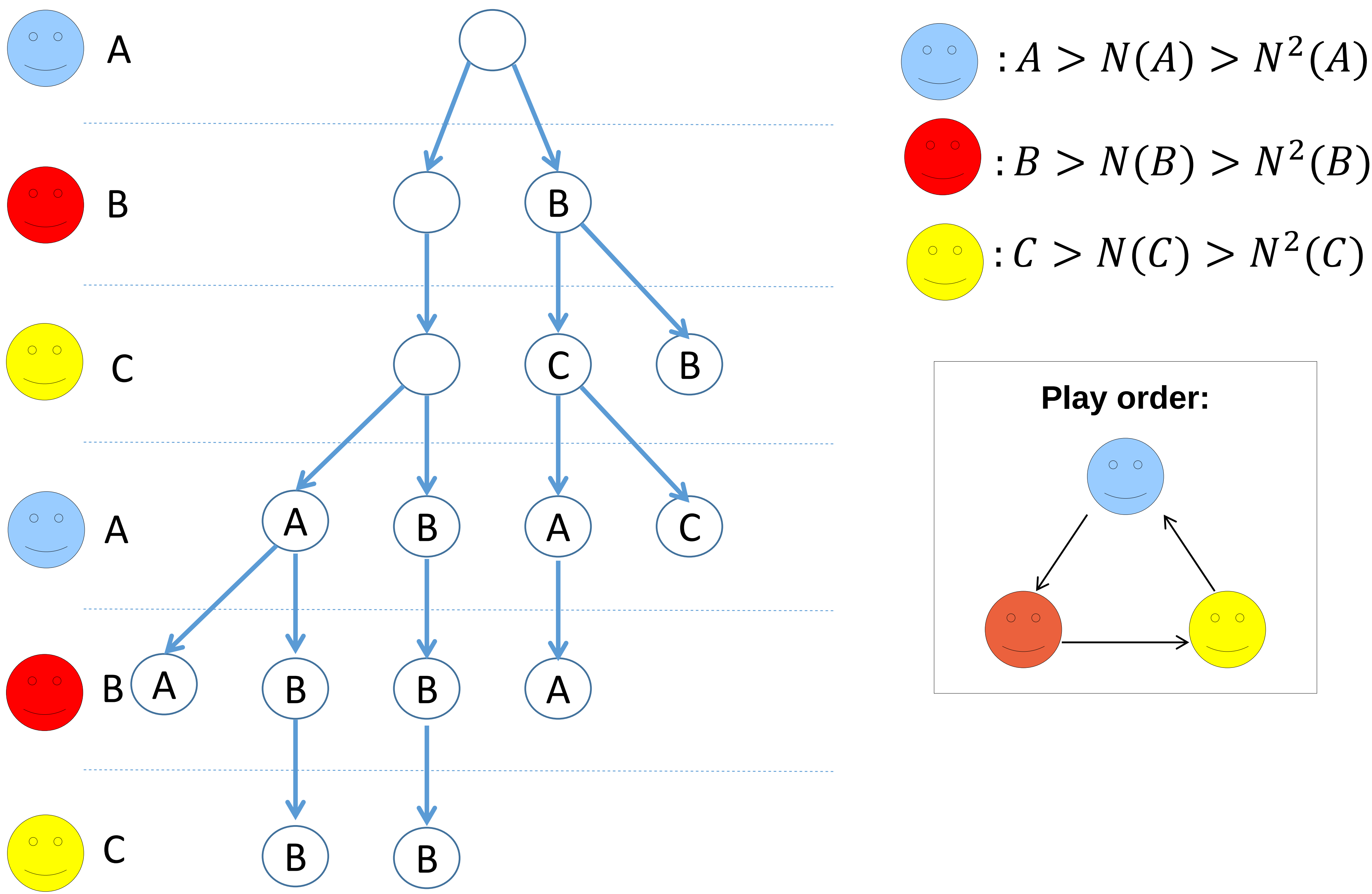

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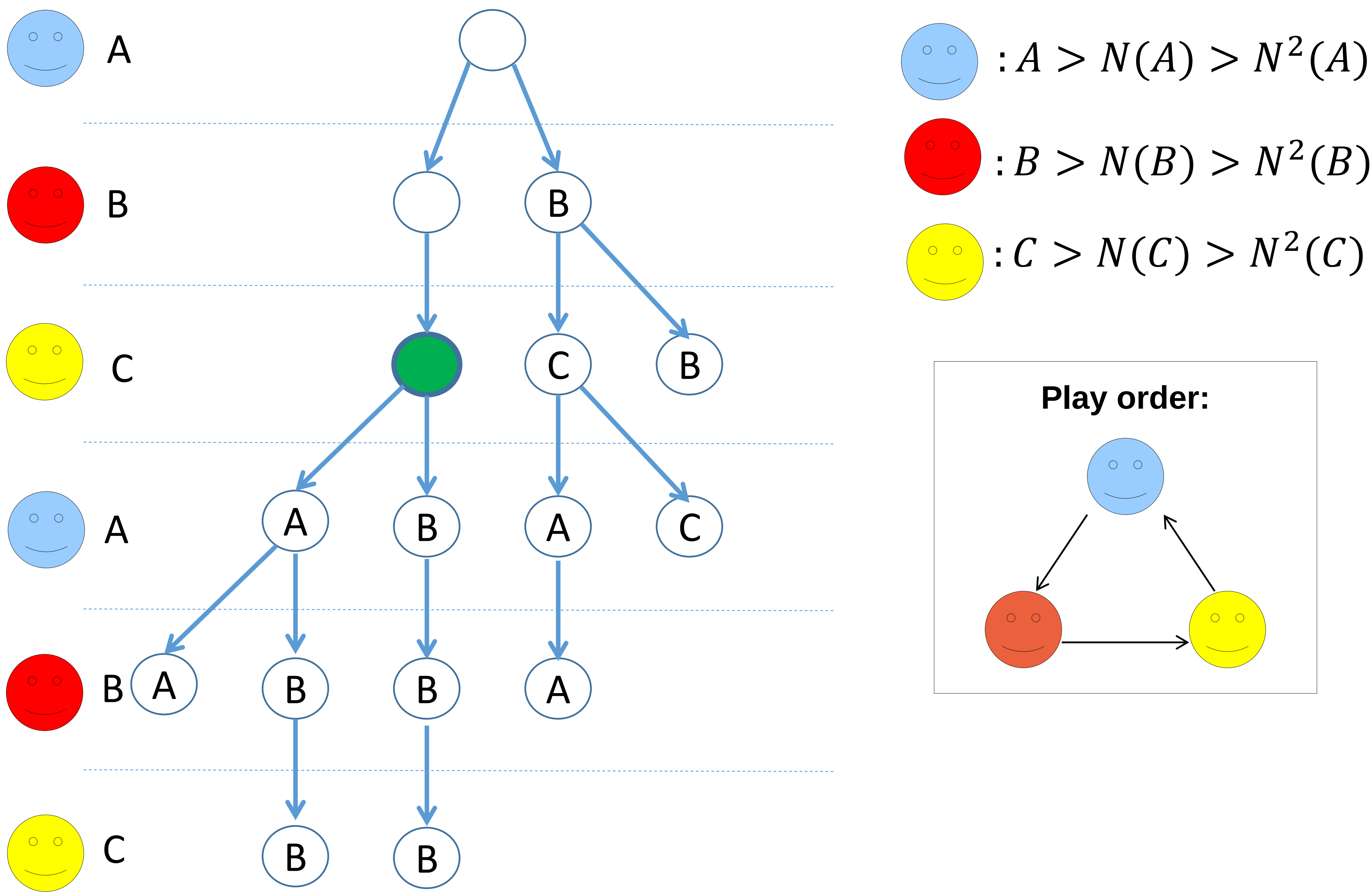

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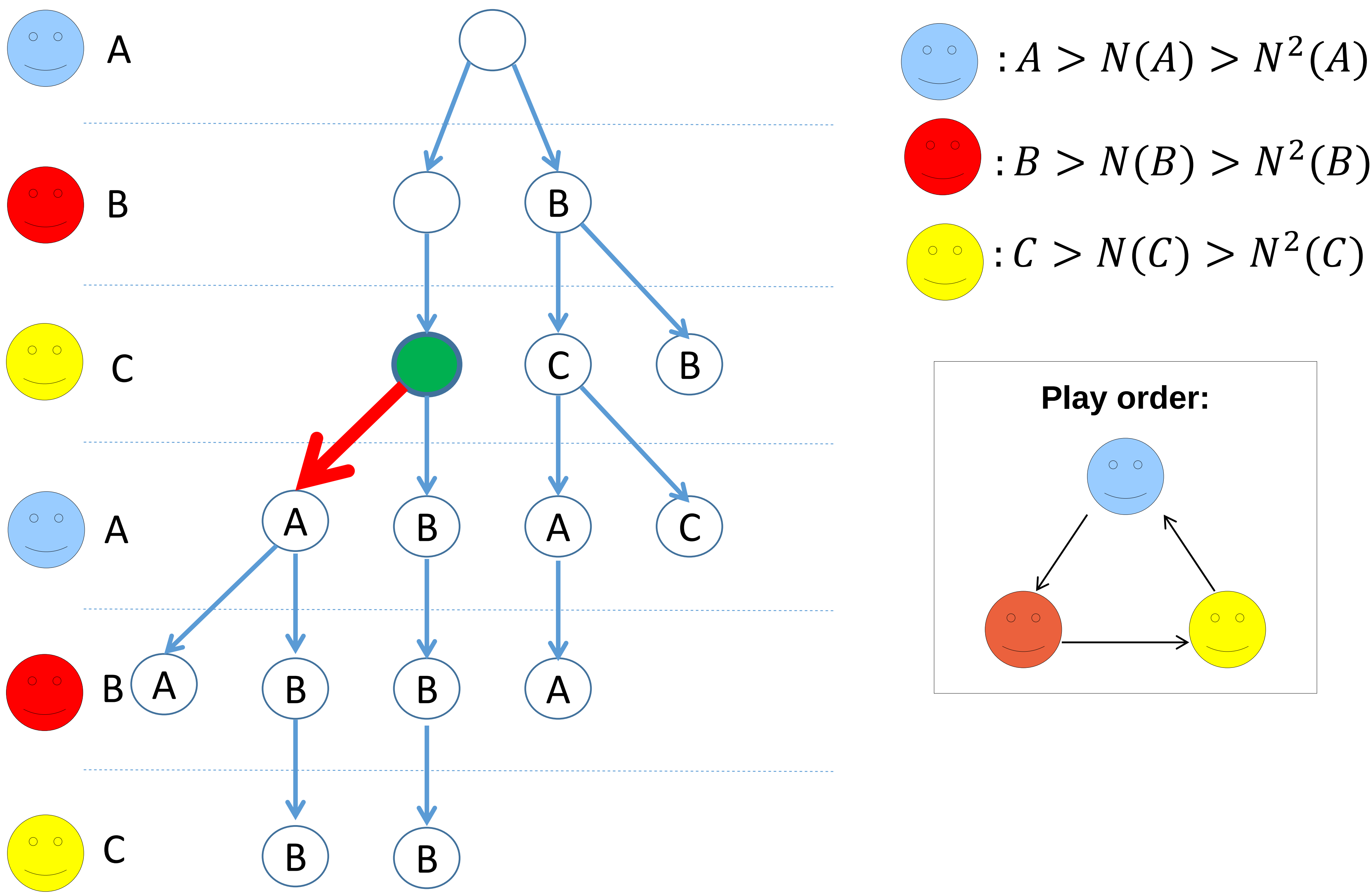

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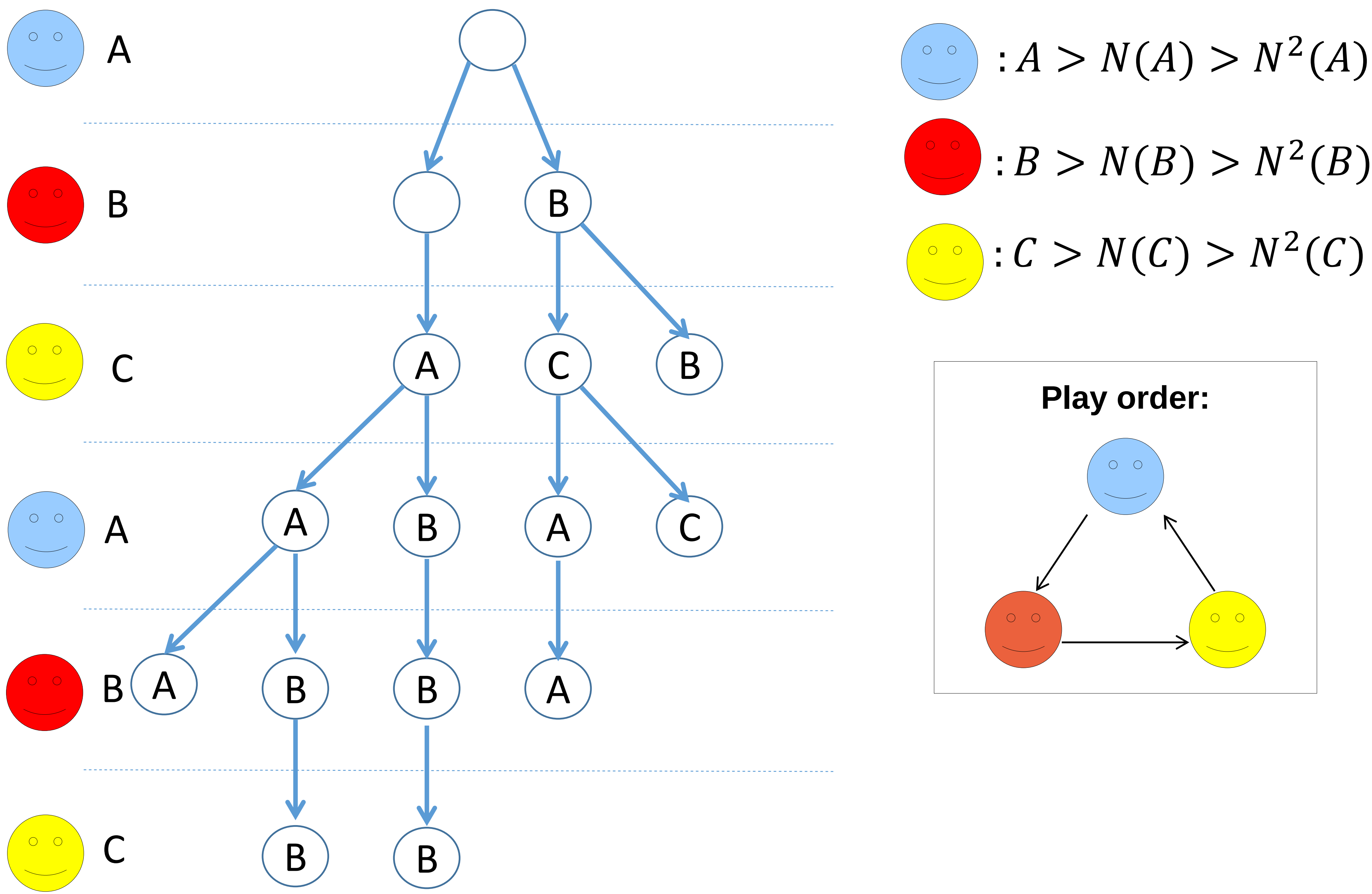
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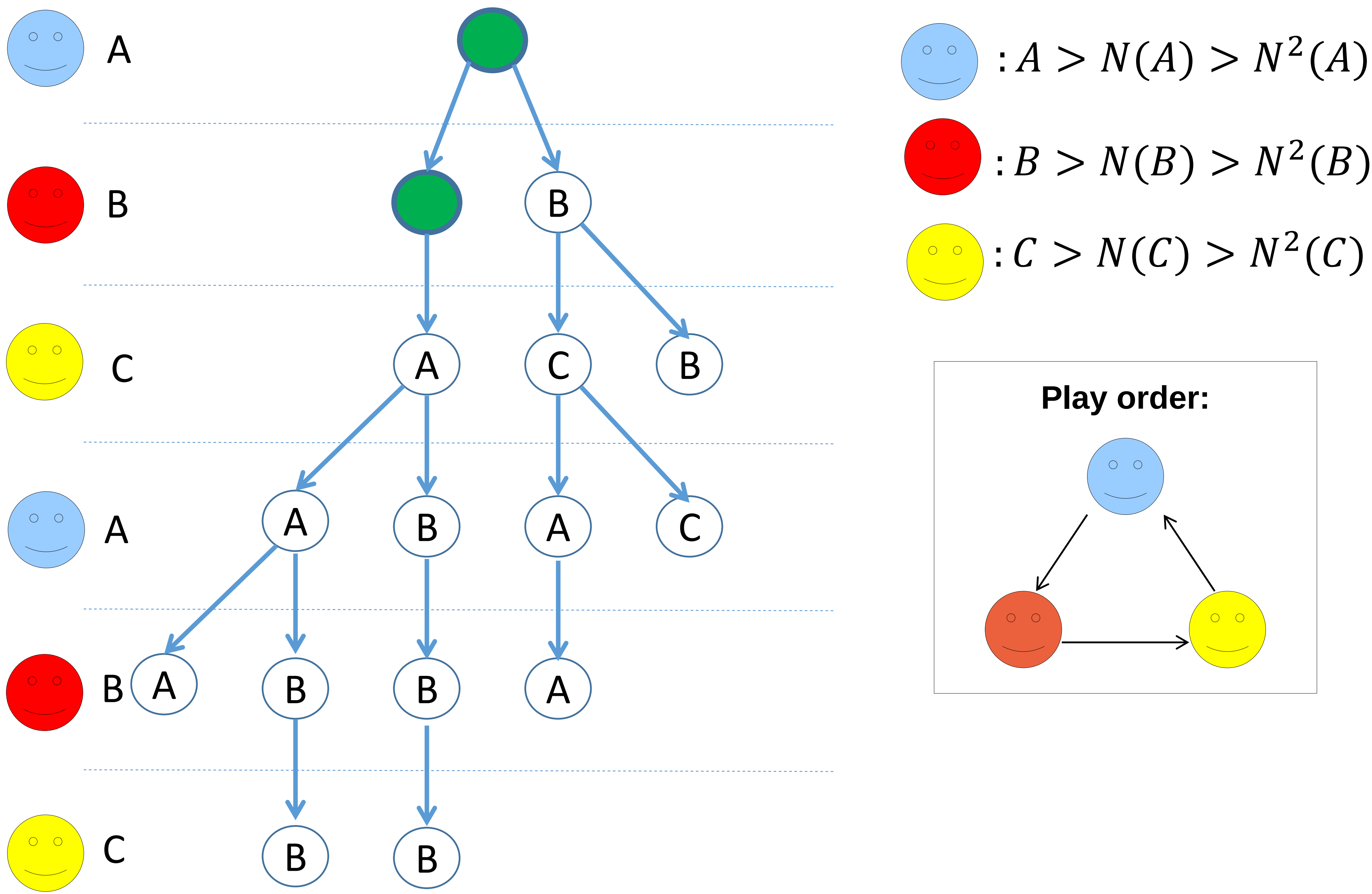

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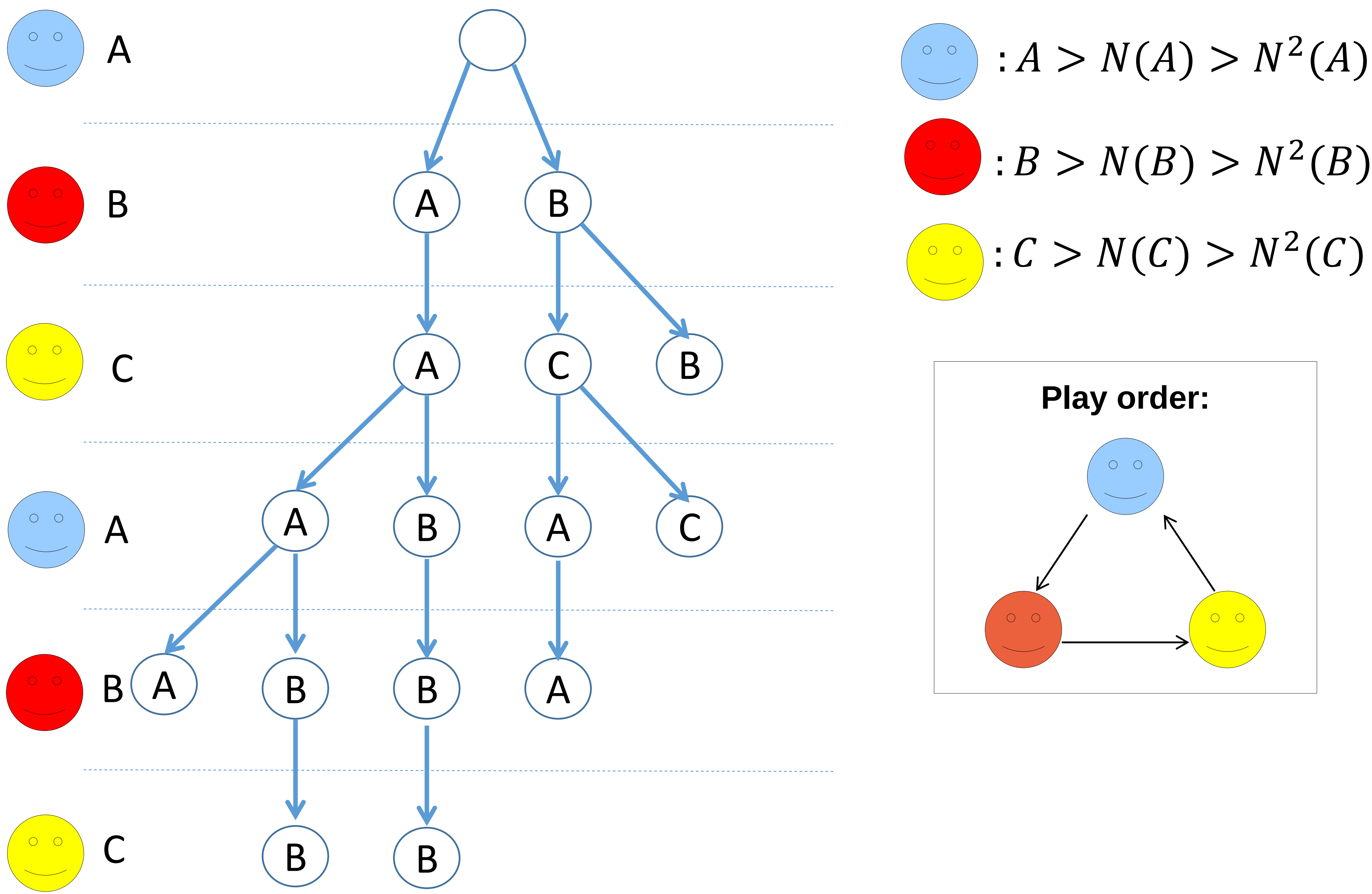
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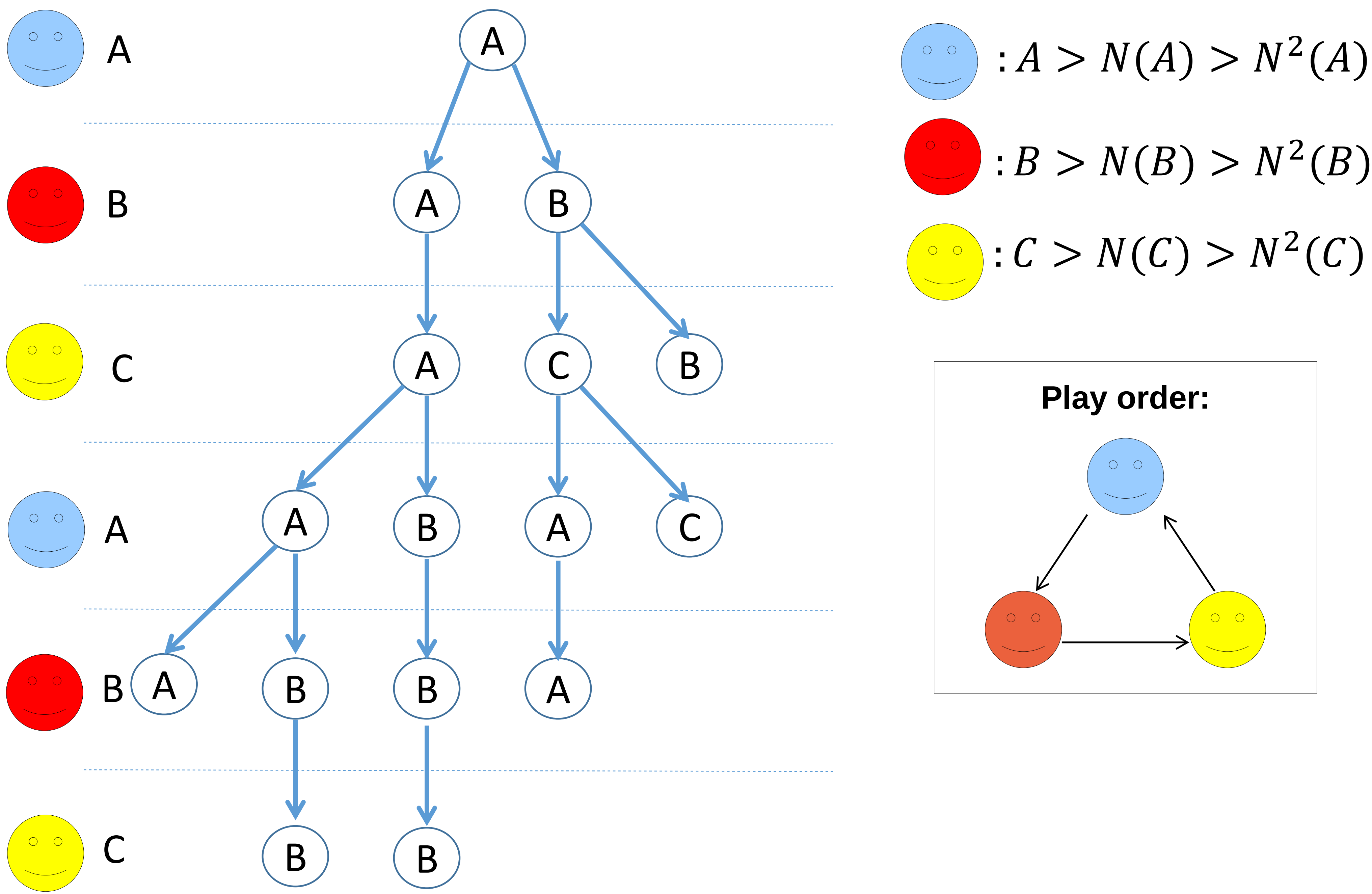
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Play order:

Definitions

Let G be a game position. Suppose that X is the first player of G. For all player X, if player $N^{i-1}(X)$ moves last, then G is called an i-position.

Generalized NIM Sum: $\oplus_m$

$$n_1 \oplus_m n_2 \oplus_m \dots \oplus_m n_k$$

Example: $3 \oplus_3 15 \oplus_3 13 \oplus_3 11$

3

15

13

11

$3 \oplus_3 15 \oplus_3 13 \oplus_3 11$

Generalized NIM Sum: $\oplus_m$

$$n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k$$

Example: $3 \oplus_3 15 \oplus_3 13 \oplus_3 11$

$3 \rightarrow 0011$

$15 \rightarrow 1111$

$13 \rightarrow 1101$

$11 \rightarrow 1011$

$3 \oplus_3 15 \oplus_3 13 \oplus_3 11$

Generalized NIM Sum: $\oplus_m$

$$n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k$$

Example: $3 \oplus_3 15 \oplus_3 13 \oplus_3 11$

3 → 0011

15 → 1111

13 → 1101

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Generalized NIM Sum: $\oplus_m$

$$n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k$$

Example: $3 \oplus_3 15 \oplus_3 13 \oplus_3 11$

3 → 0011

15 → 1111

13 → 1101

11 → 1011

$3 \oplus_3 15 \oplus_3 13 \oplus_3 11$ → "0201"

m-player normal NIM

If for all player X, her preference order is

$$X > N(X) > \cdots > N^{m-1}(X),$$

then NIM position is a 0–position(m–position) if and only if

$$n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = "00 \ldots 00"$$

※Note that this result includes the theory of two-player normal play.

S.-Y Robert Li. N-person Nim and N-person Moore's Games. Internat. J. Game Theory, Vol. 7, No. 1, pp.31-36, 1978.

New result

When does worst player take last stone?

normal play:

$$n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$$

misère play:

$$\begin{cases} n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0 (\exists n_i > 1) \\ n_1 \oplus n_2 \oplus \cdots \oplus n_k = 1 (\forall n_i \leq 1) \end{cases}$$

$m = 2$

m-player normal play:

$$n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = "00 \ldots 00"$$

When does worst player take last stone?

normal play:

$$n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$$

misère play:

$$\begin{cases} n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0 (\exists n_i > 1) \\ n_1 \oplus n_2 \oplus \cdots \oplus n_k = 1 (\forall n_i \leq 1) \end{cases}$$

$m = 2$

m-player normal play:

$$n_1 \oplus_m n_2 \oplus_m \dots \oplus_m n_k = \text{"}00 \dots 00\text{"}$$

m-player misère play:

New result

New result: m-player misère play

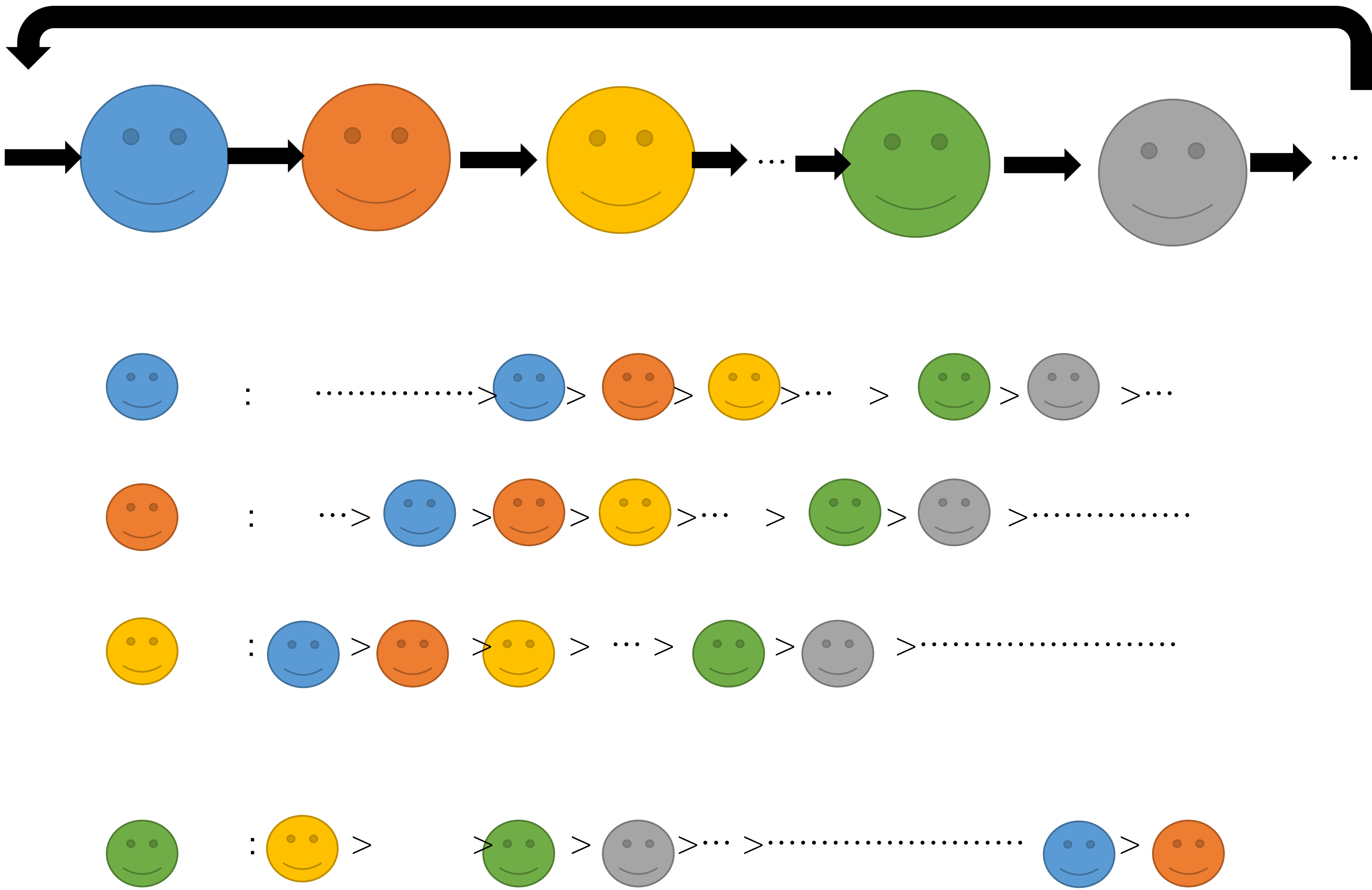
Theorem:

Assume that for all integer j and for all player X, her preference order is

$N^j(X) > N^{j+1}(X) > \cdots > N^{m-1}(X) > X > N(X)$ ... $> N^{j-1}(X)$,

then $(n_1, n_2, \ldots, n_{k-1}, n_k)$ is a j-position if and only if

$$\begin{cases} n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = "00\ldots00" (\exists n_i > 1) \\ n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = "00\ldots0j" (\forall n_i \leq 1) \end{cases}$$



New result: m-player misère play

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Assume that for all integer j and for all player X, her preference order is

$N^j(X) > N^{j+1}(X) > \cdots > N^{m-1}(X) > X > N(X)$ ... $> N^{j-1}(X)$,

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$$\begin{cases} n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = \text{"}00\ldots00\text{"} (\exists n_i > 1) \\ n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = \text{"}00\ldots0j\text{"} (\forall n_i \leq 1) \end{cases}$$

This result includes two-player misère NIM by $m = 2$ and $j = 1$

Two-player misère NIM

Theorem:

Assume that for all integer j and for all player X, her preference order is

$N^{j}(X) > N^{j+1}(X) > \cdots > N^{m-1}(X) > X > N(X)$ ... $> N^{j-1}(X)$,

then $(n_1, n_2, \ldots, n_{k-1}, n_k)$ is a j-position if and only if

$$\begin{cases} n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = \text{"}00\ldots00\text{"} (\exists n_i > 1) \\ n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = \text{"}00\ldots0j\text{"} (\forall n_i \leq 1) \end{cases}$$

Two-player misère NIM

Theorem:

Assume that for all player X, her preference order is

$$N^{1}(X) > X$$

then $(n_1, n_2, \ldots, n_{k-1}, n_k)$ is a 1-position if and only if

$$\begin{cases} n_1 \oplus_2 n_2 \oplus_2 \ldots \oplus_2 n_k = \text{"}00\ldots00\text{"} (\exists n_i > 1) \\ n_1 \oplus_2 n_2 \oplus_2 \ldots \oplus_2 n_k = \text{"}00\ldots01\text{"} (\forall n_i \leq 1) \end{cases}$$

This result also includes multiplayer normal NIM by $j = 0$

Multiplayer normal NIM

Theorem:

Assume that for all integer j and for all player X, her preference order is

$N^j(X) > N^{j+1}(X) > \cdots > N^{m-1}(X) > X > N(X)$ ... $> N^{j-1}(X)$,

then $(n_1, n_2, \ldots, n_{k-1}, n_k)$ is a j-position if and only if

$$\begin{cases} n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = \text{"}00\ldots00\text{"} (\exists n_i > 1) \\ n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = \text{"}00\ldots0j\text{"} (\forall n_i \leq 1) \end{cases}$$

Multiplayer normal NIM

Theorem:

Assume that for all player X, her preference order is

$$X > N(X) > \cdots > N^{m-1}(X),$$

then $(n_1, n_2, \ldots, n_{k-1}, n_k)$ is a 0-position if and only if

$$\begin{cases} n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = "00\ldots00"(\exists n_i > 1) \\ n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = "00\ldots00"(\forall n_i \leq 1) \end{cases}$$

When does worst player take last stone?

normal play:

$$n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$$

misère play:

$$\begin{cases} n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0 (\exists n_i > 1) \\ n_1 \oplus n_2 \oplus \cdots \oplus n_k = 1 (\forall n_i \leq 1) \end{cases}$$

$m = 2$

$m = 2$
$j = 1$

m-player normal play:

$$n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = \text{"}00 \ldots 00\text{"}$$

m-player misère play:

$$\begin{cases} n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = \text{"}00 \ldots 00\text{"} (\exists n_i > 1) \\ n_1 \oplus_m n_2 \oplus_m \ldots \oplus_m n_k = \text{"}00 \ldots 0j\text{"} (\forall n_i \leq 1) \end{cases}$$

$j = 0$

Another theorem

Theorem:

Assume that for all integer j and for each player X, her preference order is

$N^j(X) > N^{j-1}(X) > \cdots > N(X) > X > N^{m-1}(X)$ ... $> N^{j+1}(X)$,

then for all integer $n_1, n_2, \ldots, n_{k-1}$, there is an exactly one integer n_k such that NIM position $(n_1, n_2, \ldots, n_{k-1}, n_k)$ is a j-position.

Future problems

1. Another preferences

2. Another games
    1. Moore’s game, LIM, WYTHOFF, Graph Games,…