Computing the Shapley value of graph games with restricted coalitions

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October 25, 2017

GAG Workshop, Lyon, France.
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Outline

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Introduction

Graph games on a product of chains

Classical cooperative games
Restricted cooperation

Computing the Shapley value of graph games with restricted coalitions
A cooperative game is a pair \((N, \nu)\) where

i) \(N\) is a finite set of players.

ii) \(\nu : 2^N \rightarrow \mathbb{R}\) is a function with \(\nu(\emptyset) = 0\).

**Question**

How the players will share the value \(\nu(N)\)?
An answer: the Shapley value [L.S. Shapley 1953]

$$\varphi_i = \sum_{S \ni i} \frac{(|S| - 1)! \cdot (n - |S|)!}{n!} [v(S) - v(S \setminus \{i\})]$$

The vector $\varphi$ is called the Shapley value of the game $(N, v)$.

The Shapley value was obtained by imposing a set of axioms that the solution must satisfy: efficiency, linearity, symmetry, null player.
A problem

I practice not all coalitions are feasible: language barriers, geography, hierarchies.

Implicational systems

Let $\Sigma = \{A_1 \rightarrow a_1, \ldots, A_m \rightarrow a_m\}$ be an implicational system on $N$ and $X \subseteq N$. The $\Sigma$-closure of $X$, denoted $X^\Sigma$, is the smallest set containing $X$ and satisfying: $\forall 1 \leq j \leq m, A_j \subseteq X^\Sigma \Rightarrow a_j \in X^\Sigma$.

The set $F_\Sigma = \{X^\Sigma, X \subseteq N\}$ is a closure system (closed under intersection and containing $N$) and hence is a lattice (a partially ordered set where any two elements have a least upper bound and a greatest lower bound).

Example

$$\Sigma = \{2 \rightarrow 1, 4 \rightarrow 3, 6 \rightarrow 5\}$$
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For a maximal chain \( c \) and \( i \in N \), we denote by \( F(c, i) \) the last coalition in \( c \) that doesn’t contain the player \( i \), and by \( F^+(c, i) \) the first coalition in \( c \) that contains the player \( i \).

\[
\varphi_i(v) = \frac{1}{|Ch|} \sum_{c \in Ch} \frac{v(F^+(c, i)) - v(F(c, i))}{|F^+(c, i) \setminus F(c, i)|}.
\]
Define the set

$$\mathcal{A}_i = \{(F, F') \in \mathcal{F}_\Sigma^2 \mid \exists c \in Ch : F = F(c, i) \text{ and } F' = F^+(c, i)\}.$$  

For any $F \in \mathcal{F}_\Sigma$, we denote by $Ch^\downarrow(F)$ (resp. $Ch^\uparrow(F)$) the number of maximal chains of the sublattice $[\emptyset, F]$ (resp. $[F, N]$). With this notation, equation (1) becomes

$$\varphi_i(v) = \frac{1}{Ch^\downarrow(N)} \sum_{(F, F') \in \mathcal{A}_i} \frac{Ch^\downarrow(F) \cdot Ch^\uparrow(F')}{|F' \setminus F|} (v(F') - v(F)). \quad (2)$$
Introduction

Classical cooperative games
Restricted cooperation

Graph games on a product of chains
We have a partial order $(P, \preceq)$ on $N$, which is the disjoint union de chains of the same length.

\[ i \to j \in \Sigma \iff i \preceq j \]

$\mathcal{F}_\Sigma$ is isomorphic to the product of the chains of the order $(P, \preceq)$.
Graph games

The model of weighted graph games captures the interactions between pairs of players. This is done by considering an undirected graph $G = (N, E)$ with an integer weight $v_{ij}$ for each edge $\{i, j\} \in E$. We define a cooperative game $(N, \Sigma, v)$ by:

$$v(S) = \sum_{\{i,j\} \subseteq S} v_{ij} \quad \forall S \in \mathcal{F}_\Sigma.$$
Idea
Partition $\mathcal{A}_i$ in such a way that $Ch^\downarrow(F) \cdot Ch^\uparrow(F^+)$ is constant inside each block of the partition.

Proposition 3
let $i \in N$ and $c(i)$ the chain containing $i$ in $P$. The elements $\mathcal{A}_i$ are exactly the pairs $(F \cup \{i\}^\Sigma \setminus \{i\}, F \cup \{i\}^\Sigma)$ where $F \in \mathcal{F}_\Sigma$ with $F \cap c(i) = \emptyset$.

The set $\mathcal{A}_i$ can thus be identified with

$$\tilde{\mathcal{A}}_i = \{F \in \mathcal{F}_\Sigma \mid F \cap c(i) = \emptyset\}.$$

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**The partition**

We define an equivalence relation $\mathcal{R}_i$ over $\tilde{\mathcal{A}}_i$ as follows:

$$F_1 \mathcal{R}_i F_2 \iff P_{|F_1}$ is isomorphic to $P_{|F_2}.$$

**Encoding the equivalence classes**

The next proposition gives an encoding of the class $\overline{F}$, with $|F| = k$, by a vector of integers in the set:

$$\mathcal{D}_k = \{(x_0, \ldots, x_I) \in \mathbb{N}^{I+1}, \text{ such that } \sum_{t=0}^{I} x_t = m - 1, \sum_{t=0}^{I} t \cdot x_t = k\}.$$
proposition 4

Let \( i \in \mathbb{N} \). The sets \( Q_i \) and \( \mathcal{E} = \bigcup_{k=0}^{n-l} \mathcal{D}_k \) are in bijection by the mapping \( \psi : Q_i \rightarrow \mathcal{E}, \bar{F} \mapsto \psi(\bar{F}) = (x_0, \ldots, x_l) \) where \( x_t \) is the number of chains of size \( t \) in \( P|F \) for \( 1 \leq t \leq l \), and \( x_0 = m - 1 - \sum_{t=1}^{l} x_t \).

Furthermore, we have \( \psi(\bar{F}) \in \mathcal{D}_k \) with \( k = |F| \).

Proposition 5

We have \(|D_k| \in O(k^l)|.\)
**Notation**

Let $x \in \mathcal{E}$ and denote by $\mathcal{A}_i^x$ the class $\psi^{-1}(x)$.

**Lemma 1**

Assume that all the chains of $P$ have the same length and let $x \in \mathcal{E}$. Then for all $F_1, F_2 \in \mathcal{A}_i^x$, we have:

$$Ch^\downarrow(F_1 \cup \{i\}^\Sigma \setminus \{i\}) \cdot Ch^\uparrow(F_1 \cup \{i\}^\Sigma) = Ch^\downarrow(F_2 \cup \{i\}^\Sigma \setminus \{i\}) \cdot Ch^\uparrow(F_2 \cup \{i\}^\Sigma).$$
**Notation**

Pour $F \in A_i^x$:

\[
\alpha_x = \text{Ch}^\downarrow(F \cup \{i\}^\Sigma \setminus \{i\}) \cdot \text{Ch}^\uparrow(F \cup \{i\}^\Sigma)
\]

**Lemma 2**

Let $x \in \mathcal{E}$ and $k = \sum_{t=0}^{l} t \cdot x_t$. We have

\[
\alpha_x = \frac{(k + h(i))! \cdot (n - k - h(i) - 1)!}{h(i)! \cdot (l - h(i) - 1)! \cdot \prod_{t=0}^{l} [t! \cdot (l - t)!]^{x_t}}
\]
Proposition 6

Let \((N, \Sigma, \nu)\) be a weighted graph game and \(i \in N\). We have,

\[
\varphi_i(\nu) = \frac{1}{Ch(N)} \sum_{k=0}^{n-l} \sum_{x \in D_k} \sum_{j \neq i} \beta_{ij}^x \cdot \alpha_x \cdot \nu_{ij}, \quad \text{where} \quad \beta_{ij}^x = |\{F \in A_i^x \mid j \in F \cup \{i\} \Sigma\}|.
\]
Lemma 3

Let $i \neq j \in N$ and $x \in \mathcal{E}$. Then

$$\beta_{ij}^x = \begin{cases} 
0, & \text{si $j \rightarrow i$}, \\
\frac{(m-1)!}{\prod_{t=0}^{l} x_t!}, & \text{si $i \rightarrow j$}, \\
\frac{(m-2)!}{\prod_{t=0}^{l} x_t!} \cdot \sum_{t=h(j)+1}^{l} x_t, & \text{sinon}.
\end{cases}$$
Theorem 1

The Shapley value $\varphi_i$ of a player $i$ in a weighted graph game on a product of $m$ chains with the same length $l - 1$ can be computed in $O(n^{l+3})$, where $n$ is the number of players. For fixed $l$, it can be computed in polynomial time.
Thank you for your attention