Rulesets for Beatty games

Lior Goldberg   Aviezri S. Fraenkel

Games and Graphs Workshop, 2017

To appear in IJGT; online version: http://rdcu.be/wrcd

Beatty games

Any game with the following properties:

- Subtraction game with two (symmetric) piles.
- Invariant game.
- The set of P-positions is $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$, for **arbitrary** irrationals $1 < \alpha < 2 < \beta$ where $1/\alpha + 1/\beta = 1$.

Motivation: t-Wythoff

t-Wythoff ($t \in \mathbb{Z}_{\geq 1}$) is a generalization of Wythoff. It is played on two piles of tokens. Each player can either:

- Remove tokens from one pile (Nim move).
- Remove k tokens from one pile and ℓ tokens from the other, provided that $|k - \ell| < t$ (Diagonal move).

The player first unable to move loses (*normal* play).

Motivation: t-Wythoff

t-Wythoff ($t \in \mathbb{Z}_{\geq 1}$) is a generalization of Wythoff. It is played on two piles of tokens. Each player can either:

- Remove tokens from one pile (Nim move).
- Remove k tokens from one pile and ℓ tokens from the other, provided that $|k - \ell| < t$ (Diagonal move).

The player first unable to move loses (*normal* play).

Properties:

- Subtraction game with two (symmetric) piles.

Motivation: t-Wythoff

t-Wythoff ($t \in \mathbb{Z}_{\geq 1}$) is a generalization of Wythoff. It is played on two piles of tokens. Each player can either:

- Remove tokens from one pile (Nim move).
- Remove k tokens from one pile and ℓ tokens from the other, provided that $|k - \ell| < t$ (Diagonal move).

The player first unable to move loses (*normal* play).

Properties:

- Subtraction game with two (symmetric) piles.
- Invariant game.

Motivation: t-Wythoff

t-Wythoff ($t \in \mathbb{Z}_{\geq 1}$) is a generalization of Wythoff. It is played on two piles of tokens. Each player can either:

- Remove tokens from one pile (Nim move).
- Remove k tokens from one pile and ℓ tokens from the other, provided that $|k - \ell| < t$ (Diagonal move).

The player first unable to move loses (*normal* play).

Properties:

- Subtraction game with two (symmetric) piles.
- Invariant game.
- The set of P-positions is $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$ where $\alpha = [1; t, t, t, ...]$ and $\beta = \alpha + t$. Note that $1/\alpha + 1/\beta = 1$.

Beatty games

Any game with the following properties:

- Subtraction game with two (symmetric) piles.
- Invariant game.
- The set of P-positions is $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$, for **arbitrary** irrationals $1 < \alpha < 2 < \beta$ where $1/\alpha + 1/\beta = 1$.

Existence of Beatty games

Conjecture (Duchêne and Rigo, 2010)

For every irrational $1 < \alpha < 2$ and β such that $1/\alpha + 1/\beta = 1$, there exists a Beatty game.

Existence of Beatty games

Conjecture (Duchêne and Rigo, 2010)

For every irrational $1 < \alpha < 2$ and β such that $1/\alpha + 1/\beta = 1$, there exists a Beatty game.

Proof (Larsson et al., 2011)

A ruleset can be constructed by applying the $\star$-operator to the set of P-positions – taking the P-positions of the game whose moves are $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 1}\}$.

Existence of Beatty games

Conjecture (Duchêne and Rigo, 2010)

For every irrational $1 < \alpha < 2$ and β such that $1/\alpha + 1/\beta = 1$, there exists a Beatty game.

Proof (Larsson et al., 2011)

A ruleset can be constructed by applying the $\star$-operator to the set of P-positions – taking the P-positions of the game whose moves are $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 1}\}$.

Problem

This ruleset is not an explicit "one-line" ruleset (compare, for example, to Wythoff).

A ruleset for an arbitrary α

Theorem

Assume $\alpha < 1.5$. The following ruleset is a Beatty game for α:

- *Nim moves.*
- *Remove k tokens from one pile and ℓ tokens from the other, provided that $|k - \ell| < \lfloor \beta \rfloor - 1$. Except for the move $(2, \lfloor \beta \rfloor)$.*
- *Remove $\lfloor \alpha n \rfloor$ tokens from one pile and $\lfloor \beta n \rfloor - 1$ tokens from the other ($n \in \mathbb{Z}_{\geq 1}$).*
- *A* **finite** *set of additional moves.*

For $1.5 < \alpha < 2$ the ruleset is slightly more complicated.

Modified t-Wythoff (MTW)

The ruleset in the theorem explicitly mentions α. Can we do better?

Modified t-Wythoff (MTW)

The ruleset in the theorem explicitly mentions α. Can we do better?

There are simpler rulesets in the literature for specific values of α:

Example

1. $\alpha = [1; t, t, t, \ldots]$ (t-Wythoff).
2. $\alpha = [1; 1, k, 1, k, \ldots]$ (Duchêne and Rigo, 2010).

Modified t-Wythoff (MTW)

The ruleset in the theorem explicitly mentions α. Can we do better?

There are simpler rulesets in the literature for specific values of α:

Example

1. $\alpha = [1; t, t, t, \ldots]$ (t-Wythoff).
2. $\alpha = [1; 1, k, 1, k, \ldots]$ (Duchêne and Rigo, 2010).

Definition

(i) A ruleset is said to be *MTW* (Modified t-Wythoff) if it is a **finite** modification of t-Wythoff for some $t \in \mathbb{Z}_{\geq 1}$.
(ii) An irrational $1 < \alpha < 2$ is said to be *MTW*, if there exists an MTW ruleset for the corresponding Beatty game.

Modified t-Wythoff (MTW)

Theorem

Let $1 < \alpha < 2$ be irrational. Then, α is MTW if and only if

$$\alpha^2 + b\alpha - c = 0$$

for some $b, c \in \mathbb{Z}$ such that $b - c + 1 < 0$.

Forbidden subtractions

A move in the ruleset must not connect two P-positions.
There are two types of such forbidden subtractions: Direct and Crossed.

For example, consider two P-positions: $(4, 9)$ and $(1, 3)$.

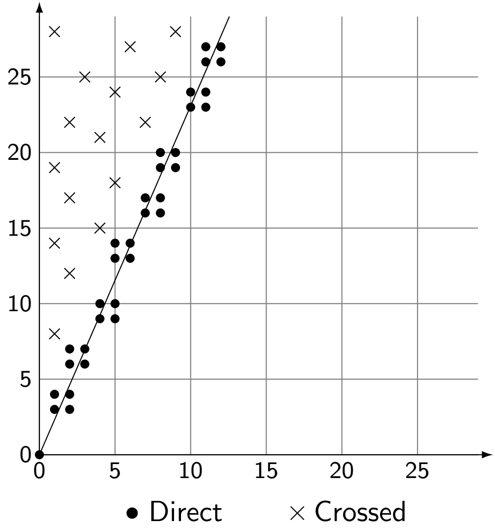
Direct

$$4 \xrightarrow{\text{Remove 3}} 1$$
$$9 \xrightarrow{\text{Remove 6}} 3$$
$$(3, 6)$$

Crossed

$$4 \xrightarrow{\text{Remove 1}} 3$$
$$9 \xrightarrow{\text{Remove 8}} 1$$
$$(1, 8)$$

Forbidden subtractions



$\alpha = [1; 2, 3, 4, \ldots]$

$\lfloor \alpha n \rfloor$	$\lfloor \beta n \rfloor$
0	0
1	3
2	6
4	9
5	13
7	16
8	19
10	23
11	26

Direct forbidden subtractions

A direct forbidden subtraction has the form:

$$(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) - (\lfloor \alpha m \rfloor, \lfloor \beta m \rfloor) = (\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b)$$

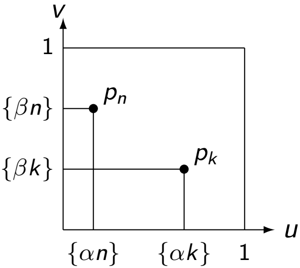
where $k = n - m$ and $a, b \in \{0, 1\}$.
The values of a and b are determined by the relative position of the points $p_k = (\{\alpha k\}, \{\beta k\})$ and $p_n = (\{\alpha n\}, \{\beta n\})$:

Direct forbidden subtractions

A direct forbidden subtraction has the form:

$$(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) - (\lfloor \alpha m \rfloor, \lfloor \beta m \rfloor) = (\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b)$$

where $k = n - m$ and $a, b \in \{0, 1\}$.
The values of a and b are determined by the relative position of the points $p_k = (\{\alpha k\}, \{\beta k\})$ and $p_n = (\{\alpha n\}, \{\beta n\})$:

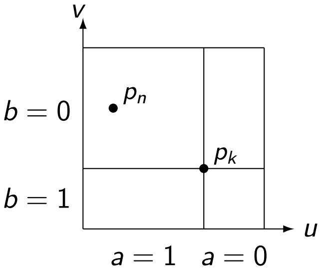


Direct forbidden subtractions

A direct forbidden subtraction has the form:

$$(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) - (\lfloor \alpha m \rfloor, \lfloor \beta m \rfloor) = (\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b)$$

where $k = n - m$ and $a, b \in \{0, 1\}$.
The values of a and b are determined by the relative position of the points $p_k = (\{\alpha k\}, \{\beta k\})$ and $p_n = (\{\alpha n\}, \{\beta n\})$:



Direct forbidden subtractions

Given $k \in \mathbb{Z}_{\geq 1}$ and $a, b \in \{0, 1\}$:
Is $(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b)$ a direct forbidden subtraction?

Direct forbidden subtractions

Given $k \in \mathbb{Z}_{\geq 1}$ and $a, b \in \{0, 1\}$:
Is $(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b)$ a direct forbidden subtraction?

Is there a point p_n in the corresponding rectangle?

Direct forbidden subtractions

Given $k \in \mathbb{Z}_{\geq 1}$ and $a, b \in \{0, 1\}$:
Is $(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b)$ a direct forbidden subtraction?

Is there a point p_n in the corresponding rectangle?

What is $\{p_n : n \in \mathbb{Z}_{\geq 0}\}$?

Direct forbidden subtractions

Given $k \in \mathbb{Z}_{\geq 1}$ and $a, b \in \{0, 1\}$:
Is $(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b)$ a direct forbidden subtraction?

Is there a point p_n in the corresponding rectangle?

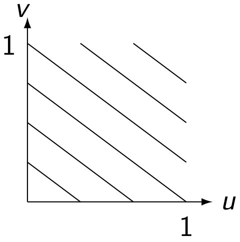
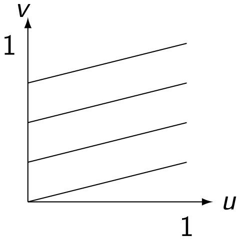
What is $\{p_n : n \in \mathbb{Z}_{\geq 0}\}$?

Easier: What is the topological closure of $\{p_n : n \in \mathbb{Z}_{\geq 0}\}$?

Direct forbidden subtractions

What is the topological closure of $\{p_n : n \in \mathbb{Z}_{\geq 0}\}$?

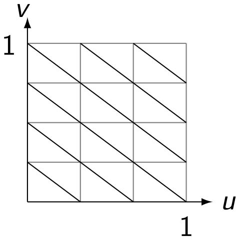
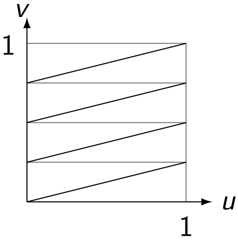
$$A\alpha + B\beta + C = 0, \quad \text{where } A, B, C \in \mathbb{Z} \quad (A > 0)$$

No solution	$B > 0$	$B < 0$
The set is dense in $[0, 1] \times [0, 1]$	 $A = 3, B = 4$	 $A = 1, B = -4$

Direct forbidden subtractions

What is the topological closure of $\{p_n : n \in \mathbb{Z}_{\geq 0}\}$?

$$A\alpha + B\beta + C = 0, \quad \text{where } A, B, C \in \mathbb{Z} \quad (A > 0)$$

No solution	$B > 0$	$B < 0$
The set is dense in $[0,1] \times [0,1]$	 $A = 3, B = 4$	 $A = 1, B = -4$

Proving the impossibility result of the MTW theorem

Observation

Let $1 < \alpha < 2$ *be irrational. Then,* α *satisfies*

$$\alpha^2 + b\alpha - c = 0, \qquad \text{where } b, c \in \mathbb{Z}, b - c + 1 < 0$$

if and only if

$$A\alpha + B\beta + C = 0, \qquad \text{where } A, B, C \in \mathbb{Z}$$

has a solution with $A = 1$ *and* $B < 0$*.*

Proving the impossibility result of the MTW theorem

Observation

Let $1 < \alpha < 2$ be irrational. Then, α satisfies

$$\alpha^2 + b\alpha - c = 0, \qquad \text{where } b, c \in \mathbb{Z}, b - c + 1 < 0$$

if and only if

$$A\alpha + B\beta + C = 0, \qquad \text{where } A, B, C \in \mathbb{Z}$$

has a solution with $A = 1$ and $B < 0$.

Case I: $A \neq 1$ and $B < 0$.
Case II: No solution or $B > 0$.

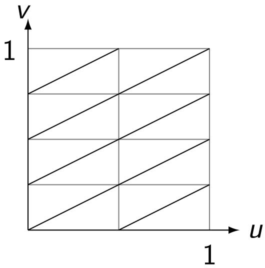
Case I: $A \neq 1$ and $B < 0$

Assume α is MTW with $A \neq 1$ and $B < 0$.

Case I: $A \neq 1$ and $B < 0$

Assume α is MTW with $A \neq 1$ and $B < 0$.

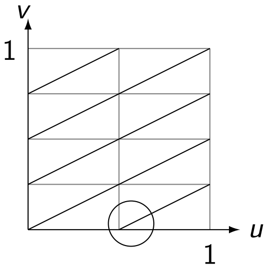
$\{p_n : n \in \mathbb{Z}_{\geq 0}\}$
$A = 2, B = -4$



Case I: $A \neq 1$ and $B < 0$

Assume α is MTW with $A \neq 1$ and $B < 0$.

$\{p_n : n \in \mathbb{Z}_{\geq 0}\}$
$A = 2, B = -4$



Take a sequence $\{n_i\}_{i=1}^{\infty}$ such that $p_{n_i} \to (1/A, 0)$.

Consider the N-positions $(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1)$.

Case I: $A \neq 1$ and $B < 0$

Nim move - not possible.

Crossed move:

$$(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1) - (\lfloor \beta m_i \rfloor, \lfloor \alpha m_i \rfloor) =$$
$$(\lfloor \alpha n_i \rfloor - \lfloor \beta m_i \rfloor, \lfloor \beta n_i \rfloor - \lfloor \alpha m_i \rfloor - 1).$$

Difference is:

$$(\lfloor \beta n_i \rfloor - \lfloor \alpha m_i \rfloor - 1) - (\lfloor \alpha n_i \rfloor - \lfloor \beta m_i \rfloor) \approx$$
$$(\beta - \alpha)(n_i + m_i) \to \infty.$$

Case I: $A \neq 1$ and $B < 0$

Nim move - not possible.

Crossed move:

$$(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1) - (\lfloor \beta m_i \rfloor, \lfloor \alpha m_i \rfloor) =$$
$$(\lfloor \alpha n_i \rfloor - \lfloor \beta m_i \rfloor, \lfloor \beta n_i \rfloor - \lfloor \alpha m_i \rfloor - 1).$$

Difference is:

$$(\lfloor \beta n_i \rfloor - \lfloor \alpha m_i \rfloor - 1) - (\lfloor \alpha n_i \rfloor - \lfloor \beta m_i \rfloor) \approx$$
$$(\beta - \alpha)(n_i + m_i) \to \infty.$$

$\Rightarrow$ At most finitely many n_i's are solved by a crossed move.

Case I: $A \neq 1$ and $B < 0$

Direct move:

$$(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1) - (\lfloor \alpha m_i \rfloor, \lfloor \beta m_i \rfloor) =$$
$$(\lfloor \alpha k_i \rfloor + a_i, \lfloor \beta k_i \rfloor + b_i - 1).$$

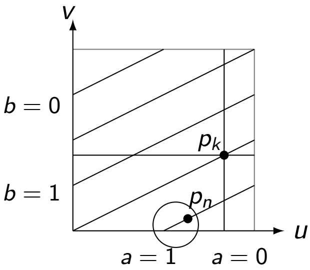
where $k_i = n_i - m_i$ and $a_i, b_i \in \{0, 1\}$.

Case I: $A \neq 1$ and $B < 0$

Direct move:

$$(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1) - (\lfloor \alpha m_i \rfloor, \lfloor \beta m_i \rfloor) =$$
$$(\lfloor \alpha k_i \rfloor + a_i, \lfloor \beta k_i \rfloor + b_i - 1).$$

where $k_i = n_i - m_i$ and $a_i, b_i \in \{0, 1\}$.



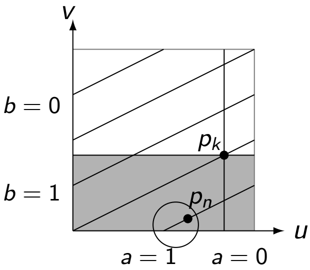
Finitely many k_i's and $\{\beta n_i\} \to 0$

Case I: $A \neq 1$ and $B < 0$

Direct move:

$$(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1) - (\lfloor \alpha m_i \rfloor, \lfloor \beta m_i \rfloor) =$$
$$(\lfloor \alpha k_i \rfloor + a_i, \lfloor \beta k_i \rfloor + b_i - 1).$$

where $k_i = n_i - m_i$ and $a_i, b_i \in \{0, 1\}$.



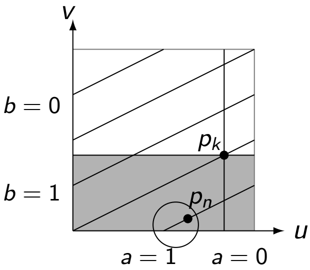
Finitely many k_i's and $\{\beta n_i\} \to 0$
$\Rightarrow$ Eventually, p_n is below p_k

Case I: $A \neq 1$ and $B < 0$

Direct move:

$$(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1) - (\lfloor \alpha m_i \rfloor, \lfloor \beta m_i \rfloor) =$$
$$(\lfloor \alpha k_i \rfloor + a_i, \lfloor \beta k_i \rfloor + b_i - 1).$$

where $k_i = n_i - m_i$ and $a_i, b_i \in \{0, 1\}$.



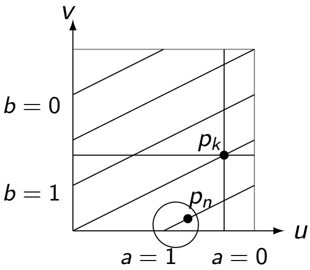
Finitely many k_i's and $\{\beta n_i\} \to 0$
$\Rightarrow$ Eventually, p_n is below p_k
$\Rightarrow$ $b_i = 1$

Case I: $A \neq 1$ and $B < 0$

Direct move:

$$(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1) - (\lfloor \alpha m_i \rfloor, \lfloor \beta m_i \rfloor) =$$
$$(\lfloor \alpha k_i \rfloor + a_i, \lfloor \beta k_i \rfloor + b_i - 1).$$

where $k_i = n_i - m_i$ and $a_i, b_i \in \{0, 1\}$.



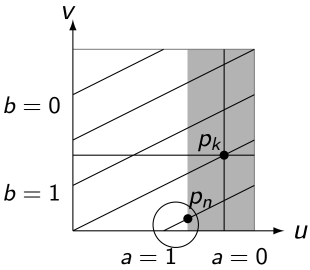
Finitely many k_i's and $\{\beta n_i\} \to 0$
$\Rightarrow$ Eventually, p_n is below p_k
$\Rightarrow b_i = 1$
$\Rightarrow a_i = 1$

Case I: $A \neq 1$ and $B < 0$

Direct move:

$$(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1) - (\lfloor \alpha m_i \rfloor, \lfloor \beta m_i \rfloor) =$$
$$(\lfloor \alpha k_i \rfloor + a_i, \lfloor \beta k_i \rfloor + b_i - 1).$$

where $k_i = n_i - m_i$ and $a_i, b_i \in \{0, 1\}$.



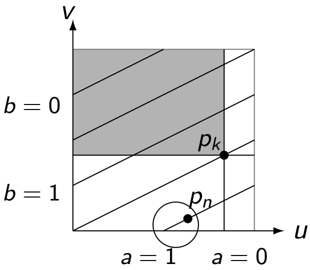
Finitely many k_i's and $\{\beta n_i\} \to 0$
$\Rightarrow$ Eventually, p_n is below p_k
$\Rightarrow b_i = 1$
$\Rightarrow a_i = 1$
$\Rightarrow 1/A < \{\alpha n_i\} < \{\alpha k_i\}$.

Case I: $A \neq 1$ and $B < 0$

Direct move:

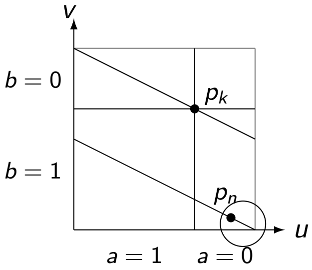
$$(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1) - (\lfloor \alpha m_i \rfloor, \lfloor \beta m_i \rfloor) = (\lfloor \alpha k_i \rfloor + a_i, \lfloor \beta k_i \rfloor + b_i - 1).$$

where $k_i = n_i - m_i$ and $a_i, b_i \in \{0, 1\}$.



Finitely many k_i's and $\{\beta n_i\} \to 0$
$\Rightarrow$ Eventually, p_n is below p_k
$\Rightarrow b_i = 1$
$\Rightarrow a_i = 1$
$\Rightarrow 1/A < \{\alpha n_i\} < \{\alpha k_i\}$.
The move is $(\lfloor \alpha k_i \rfloor + 1, \lfloor \beta k_i \rfloor)$ which is a forbidden subtraction.

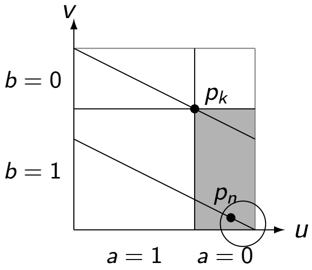
Case II: No solution or $B > 0$



Take a sequence $\{n_i\}_{i=1}^{\infty}$ such that $p_{n_i} \to (1, 0)$.

As before, we have to consider the move: $(\lfloor \alpha k_i \rfloor + a_i, \lfloor \beta k_i \rfloor + b_i - 1)$.

Case II: No solution or $B > 0$

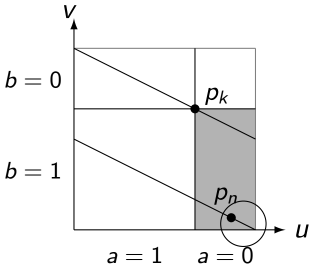


Take a sequence $\{n_i\}_{i=1}^{\infty}$ such that $p_{n_i} \to (1, 0)$.

As before, we have to consider the move: $(\lfloor \alpha k_i \rfloor + a_i, \lfloor \beta k_i \rfloor + b_i - 1)$.

Eventually, $a_i = 0$ and $b_i = 1$.

Case II: No solution or $B > 0$



Take a sequence $\{n_i\}_{i=1}^{\infty}$ such that $p_{n_i} \to (1, 0)$.

As before, we have to consider the move: $(\lfloor \alpha k_i \rfloor + a_i, \lfloor \beta k_i \rfloor + b_i - 1)$.

Eventually, $a_i = 0$ and $b_i = 1$.
This is impossible as this move is a P-position.

Questions?