Nordhaus-Gaddum inequalities for coloring games

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Joint work with
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Definitions

Proper coloring

A coloring of a graph is the assignment of a color to each vertex of the graph. A coloring is proper if two adjacent vertices have different colors. The chromatic number of a graph $G$ is denoted by $\chi(G)$. 

![Graph with proper coloring](image-url)
The **coloring game** was introduced by Brahms in 1981 and rediscovered in 1991 by Bodlaender.

- At start: a graph $G$ uncolored and a set $\Phi$ of colors.
- Alice and Bob take turns coloring an uncolored vertex of $G$ with a color of $\Phi$.
- Alice wins when the graph is fully colored. Bob wins if he can prevent Alice’s victory.
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Example

Coloring game

A little game...
Example
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A little game…
Monotony by subset?

Question
If Alice has a winning strategy for $k$ colors on a graph $G$, does she have a winning strategy for $k$ colors on any subgraph of $G$?
Coloring game

Another game?
Coloring game

Another game?
Coloring game

Another game?
Coloring game

Another game?
Coloring game

Another game?
Coloring game

Another game ?
Coloring game

Another game?
Coloring game

Another game?
Coloring game

Another game?
Monotony by number of colors?
An open problem

Let $G$ be a graph on which Alice has a winning strategy with $k$ colors.

Let $k' > k$.

Does Alice have a winning strategy for $G$ with $k'$ colors?
Monotony by number of colors? 
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Does Alice have a winning strategy for $G$ with $k'$ colors?
The **game chromatic number** of a graph $G$ is the smaller number of colors for which Alice has a winning strategy for $G$.

**Trivial bounds**

For any graph $G$, $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$. 
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Theorem [Nordhaus and Gaddum, 1956]

For any graph $G$ of order $n$, $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$.

Survey: [Aouiche and Hansen, 2013].
Norhaus-Gaddum inequalities

Result (1)

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These bounds are tight for an infinite number of values of $n$.

Theorem

- For any graph $G$ of order $n$, $2\sqrt{n} \leq \chi_g(G) + \chi_g(\overline{G}) \leq \left\lceil \frac{3n}{2} \right\rceil$.
- These bounds are asymptotically tight.

For infinite values of $n$, there are graphs $G$ such that
- $\chi_g(G) + \chi_g(\overline{G}) = 2\sqrt{2n} - 1$ and
- $\chi_g(G) + \chi_g(\overline{G}) = \left\lceil \frac{4n}{3} \right\rceil - 1$. 
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Lower bound

For any graph $G$ of order $n$, $2\sqrt{n} \leq \chi(G) + \chi(G) \leq \chi_g(G) + \chi_g(G)$.

This lemma is tight for $G = P_1$.

Consider $G$ is a complete $(\sqrt{n})$-partite graph.

$$\chi(G) + \chi(G) = 2\sqrt{n}$$

Consider $G$ is a complete $(\sqrt{\frac{n}{2}})$-partite graph.

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For any graph $G$ of order $n$, $\chi_g(G) + \chi_g(\overline{G}) \leq \lceil \frac{3n}{2} \rceil$.

**Proof**: Assume $n$ is even.

Alice colors with priority vertices with degree at least $\frac{n}{2}$.

Let $A(G)$ be the set of vertices of degree larger than $\frac{n}{2}$, $B(G) = V(G) - A(G)$.

- $A(G) = B(\overline{G})$ and $B(G) = A(\overline{G})$.
- If $|B(G)| < \lceil \frac{n}{4} \rceil$, then $\chi_g(G) \leq \frac{n}{2}$.
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Nordhaus-Gaddum inequalities

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This lemma is tight for $G = P_1$ and $G = P_4$.

Consider the joint graph $G_l = S_l + K_{\left\lceil \frac{l}{2} \right\rceil}$, $n = l + \left\lceil \frac{l}{2} \right\rceil$

- $\chi_g(G_l) = 2 \left\lceil \frac{l}{2} \right\rceil - 1$
- $\chi_g(\overline{G_l}) = l$
- If $n \geq 5$ and $n \neq 1 \text{mod} 3$, then $\chi_g(G_l) + \chi_g(\overline{G_l}) = \left\lceil \frac{4n}{3} \right\rceil - 1$. 
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Norhaus-Gaddum inequalities

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Theorem [Nordhaus and Gaddum, 1956]

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Marking game
Or "colorblind coloring game"

Marking game [Zhu, 1999]

- $G$ is a graph, $k$ an integer.
- Alice and Bob take turns marking an unmarked vertex of $G$.
- At each turn, the marked vertex must have no more than $k - 1$ marked neighbors.
- Alice wins when all the vertices are marked, Bob wins otherwise.
- The smaller $k$ for which Alice has a winning strategy on $G$ is the \textbf{coloring game number}, denoted by $\text{col}_g(G)$.

For any graph $G$, $\chi_g(G) \leq \text{col}_g(G)$. 
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For any graph \( G \), \( \chi_g(G) \leq \text{col}_g(G) \).
Marking game

A much easier game to study...
Norhaus-Gaddum inequalities

Result (2)

**Theorem**

For any graph $G$ of order $n$, $2\sqrt{n} \leq \chi_g(G) + \chi_g(\overline{G}) \leq \lceil \frac{3n}{2} \rceil$.

For infinite values of $n$, there are graphs $G$ such that

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For any graph $G$ of order $n$, $2 \left\lceil \frac{n}{2} \right\rceil \leq col_g(G) + col_g(\overline{G}) \leq \left\lceil \frac{8n-2}{5} \right\rceil$.

The lower bound is tight for infinitely many values of $n$.

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For any graph $G$ of order $n$, $2\sqrt{n} \leq \chi_g(G) + \chi_g(G) \leq \left\lceil \frac{3n}{2} \right\rceil$.

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For any graph $G$ of order $n$, $2 \left\lceil \frac{n}{2} \right\rceil \leq \text{col}_g(G) + \text{col}_g(\overline{G})$.

- If $n$ is even, $\text{col}_g(G) + \text{col}_g(\overline{G}) \leq n$.
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Consider the case $n$ is odd.

Proof (sketch) : Order the vertices by increasing degree. Bob always selects the unselected vertex with largest degree.
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When $n$ is odd, this bound is reached by $K_n$.
When $n$ is even, this bound is reached for every $n \neq 2, 4$. 
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Lemma

For any graph $G$ of order $n$, $col_g(G) + col_g(\overline{G}) \leq \left\lfloor \frac{8n-2}{5} \right\rfloor$.

Proof (sketch) : Alice always selects the unselected vertex with largest degree. Observe the vertex selected after $col_g(G) - 1$ of its neighbors.
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This lemma is tight for $G = P_4$.
Consider the joint graphs $G_l = S_l + K_{l+1}$ and $G'_l = S_l + K_{l+2}$.

- $\text{col}_g(G_l) = 2l + 1$, $\text{col}_g(G'_l) = 2l + 2$
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For any graph \( G \) of order \( n \), \( \text{col}_g(G) + \text{col}_g(\overline{G}) \leq \left\lfloor \frac{8n-2}{5} \right\rfloor \).

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**Result (3)**

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For any graph $G$ of order $n$, $2\sqrt{n} \leq \chi_g(G) + \chi_g(\overline{G}) \leq \left\lceil \frac{3n}{2} \right\rceil$.

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Open problems

"Nothing ends, Adrian. Nothing ever ends."

- Tighten those bounds?
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Merci !