Properties of Gauss digitized shapes and digital surface integration

Jacques-Olivier Lachaud
Laboratory of Mathematics
CNRS / University Savoie Mt Blanc

Boris Thibert
Laboratory Jean Kuntzman
CNRS / University J. Fourier

July 8th, 2015
Autrans, DigitalSnow

LAMA
Laboratoire de Mathématiques
Université de Savoie

UMR 5127

Properties of Gauss digitized shapes, digital surface integration

1 Context and objectives

2 Properties of Gauss digitized sets

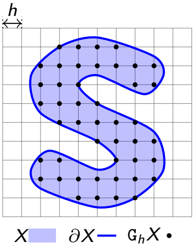
3 Manifoldness of digitized boundary

4 Injectiveness of projection

5 Digital surface integration

Properties of digitized shapes

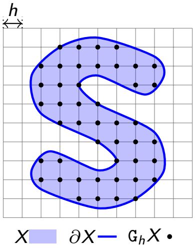
- digitization : any function that maps a subset $X \subset \mathbb{R}^d$ to a subset of $h \cdot \mathbb{Z}^d$, h is the sampling gridstep.



- Question: what are topological and geometric properties kept by digitization ?

Properties of digitized shapes

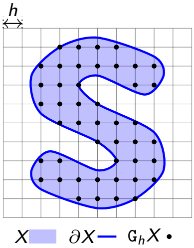
- digitization : any function that maps a subset $X \subset \mathbb{R}^d$ to a subset of $h \cdot \mathbb{Z}^d$, h is the sampling gridstep.



- Question: what are topological and geometric properties kept by digitization ?
- Specialized version of sampling problem

Properties of digitized shapes

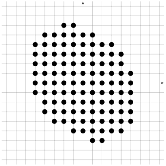
- digitization : any function that maps a subset $X \subset \mathbb{R}^d$ to a subset of $h \cdot \mathbb{Z}^d$, h is the sampling gridstep.



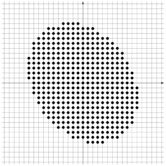
- Question: what are topological and geometric properties kept by digitization ?
- Specialized version of sampling problem
- Almost nothing is “kept”, a better word is “can be infered”.

The role of the sampling gridstep

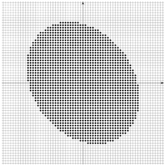
- Generally, the smaller the gridstep h the more faithful is/looks the digitization



$\mathtt{G}_h(X)$



$\mathtt{G}_{h/2}(X)$

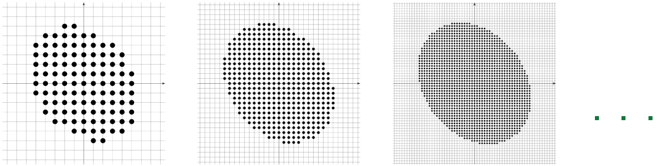


$\mathtt{G}_{h/4}(X)$

...

The role of the sampling gridstep

- Generally, the smaller the gridstep h the more faithful is/looks the digitization

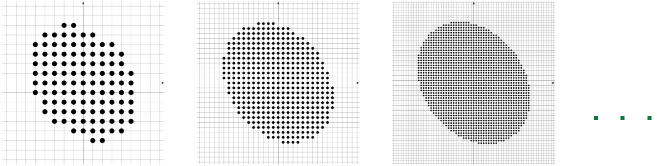


$\mathtt{G}_h(X)$ $\mathtt{G}_{h/2}(X)$ $\mathtt{G}_{h/4}(X)$ $\ldots$

- most topology preservation results are valid for specific subsets of $\mathbb{R}^d$, and for small enough gridstep.

The role of the sampling gridstep

- Generally, the smaller the gridstep h the more faithful is/looks the digitization



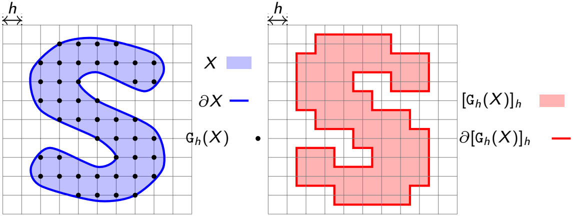
$\mathtt{G}_h(X)$ $\mathtt{G}_{h/2}(X)$ $\mathtt{G}_{h/4}(X)$ ...

- most topology preservation results are valid for specific subsets of $\mathbb{R}^d$, and for small enough gridstep.
- digital geometric quantities approach their Euclidean counterpart as the gridstep tend to zero, also for specific subsets of $\mathbb{R}^d$.
  ⇒multigrid convergence [Pavlidis 1982, Serra 1982]

Digitizations process

Definition (Gauss digitization)

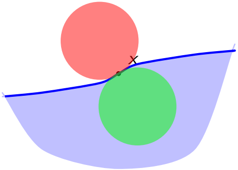
For $X \subset \mathbb{R}^d$, its Gauss digitization is $\mathtt{G}_h(X) := X \cap h \cdot \mathbb{Z}^d$.



- $[\mathtt{G}_h(X)]_h$:= union of h-cubes centered on $\mathtt{G}_h(X)$
- $\partial_h X := \partial[\mathtt{G}_h(X)]_h$:= boundary of previous set
- Many other digitization schemes: inner Jordan $\mathtt{J}^-$ and outer Jordan $\mathtt{J}^+$, Hausdorff digitizations [Ronse, Tajine 2000, Tajine, Ronse 2002]

Topology preservation of digitization

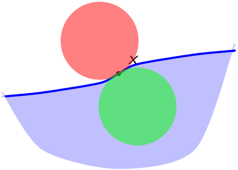
- Question: when is $\partial_h X$ homeomorphic to ∂X ?
- related to R-regularity or par(R)-regularity [Pavlidis 1982]



- 2D results for fine enough h [Stelldinger, Köthe 2005, Latecki et al. 1998]

Topology preservation of digitization

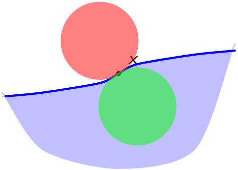
- Question: when is $\partial_h X$ homeomorphic to ∂X ?
- related to R-regularity or par(R)-regularity [Pavlidis 1982]



- 2D results for fine enough h [Stelldinger, Köthe 2005, Latecki et al. 1998]
- But false starting from 3D !

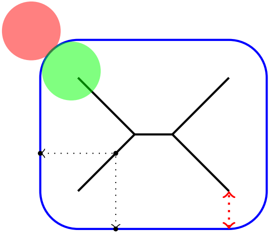
Topology preservation of digitization

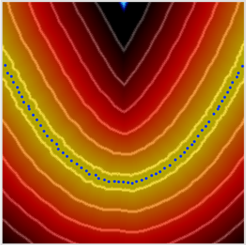
- Question: when is $\partial_h X$ homeomorphic to ∂X ?
- related to R-regularity or par(R)-regularity [Pavlidis 1982]



- 2D results for fine enough h [Stelldinger, Köthe 2005, Latecki et al. 1998]
- But false starting from 3D !
- Only homotopy preservation [Stelldinger, Köthe 2005]

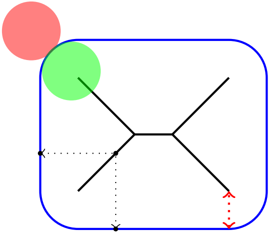
Distance and R-offset

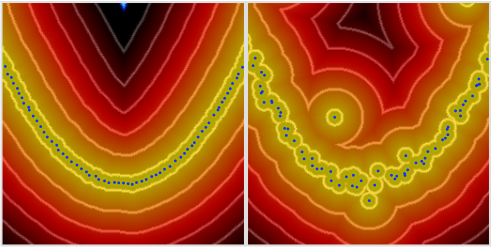




- distance d_K to a compact set K, projection ξ_K onto K
- it is Hausdorff stable whatever the dimension
- reach of ∂X := infimum of distances to medial axis.
- homotopy stability between R-offsets of X and K, if X has positive reach, K is a dense enough sampling, suitable values of R
  [Chazal, Lieutier 2008, Niyogi et al. 2008]

Distance and R-offset





- distance d_K to a compact set K, projection ξ_K onto K
- it is Hausdorff stable whatever the dimension
- reach of ∂X := infimum of distances to medial axis.
- homotopy stability between R-offsets of X and K, if X has positive reach, K is a dense enough sampling, suitable values of R
  [Chazal, Lieutier 2008, Niyogi et al. 2008]

Multigrid convergence of geometric estimators

Geometric estimator $\hat{\epsilon}$ multigrid convergent for a family of shapes $\mathbb{X}$ to a geom. quantity ϵ, $\exists h_0$, $\forall 0 < h < h_0$
$\forall X \in \mathbb{X}, |\hat{\epsilon}(\mathtt{G}_h(X)) - \epsilon(X)| \leq \tau(h)$, with $\lim_{h \to 0} \tau(h) = 0$.

- volume of a convex set X by counting [Gauss, Dirichlet]. $\tau(h) = O(h)$.
- even better bounds for C^3-smooth strictly convex X [Huxley 1990]
- volume under monotonic functions by counting (see [Krätzle 1988, Krätzle, Nowak 1991]). $\tau(h) = O(h)$.
- 2D and 3D moments of small order [Klette, Žunić 2000]
- perimeter with MLP, ϵ-sausage or DSS segmentation [Klette, Žunić 2000] [Kovalevsky, Fuchs92] [Sloboda, Zatko 1996] [Klette, Rosenfeld 2004], pattern and polygonal approximation [Tajine, Baudrier, Mazo]
- 3D area estimation, i.e. H^2: thickening [Stelldinger et al. 2007] (but see Weyl formula [Weyl 1939]), use Cauchy-Crofton integral formula [Liu et al. 2010]
- 3D local area estimation by integration of normals [Lenoir et al. 1996, Coeurjolly et al. 2003]

Multigrid convergence of geometric estimators

Geometric estimator $\hat{\epsilon}$ multigrid convergent for a family of shapes $\mathbb{X}$ to a geom. quantity ϵ, $\exists h_0$, $\forall 0 < h < h_0$
$\forall X \in \mathbb{X}, |\hat{\epsilon}(\mathtt{G}_h(X)) - \epsilon(X)| \leq \tau(h)$, with $\lim_{h\to 0} \tau(h) = 0$.

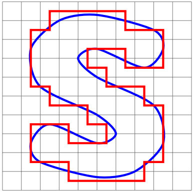
- volume of a convex set X by counting [Gauss, Dirichlet]. $\tau(h) = O(h)$.
- even better bounds for C^3-smooth strictly convex X [Huxley 1990]
- volume under monotonic functions by counting (see [Krätzle 1988, Krätzle, Nowak 1991]). $\tau(h) = O(h)$.

Theorem

Let X be a compact domain of $\mathbb{R}^d$ such that the reach of ∂X is greater than ρ, and $h < \frac{\rho}{\sqrt{d}}$. Let $\mathtt{D}$ be any digitization such that $\mathtt{J}^-_h(X) \subset \mathtt{D}_h(X) \subset \mathtt{J}^+_h(X)$. Digital and continuous volumes follows

$$\left|\mathrm{Vol}(X) - \widehat{\mathrm{Vol}}(\mathtt{D}_h(X), h)\right| \leq 2^{d+1}\sqrt{d}\,\mathrm{Area}(\partial X) h\,. \tag{1}$$

Multigrid convergence of local geometric estimators



- slight difficulty to define it: must relate ∂X with $\partial_h X$
- 2D tangent/normal estimation: MDSS [de Vieilleville et al. 2007, Lachaud et al. 2007], polynomial fitting [Provot, Gérard 2011], binomial convolution [Esbelin, Malgouyres 2009, Esbelin et al. 2011]
- 2D and 3D normals, mean and principal curvatures with integral invariants [Coeurjolly et al. 2013, Coeurjolly et al. 2014]
- nD normals with Voronoi Covariance Measure [Cuel et al. 2014]
- stability of curvature measures [Chazal,Cohen-Steiner,Lieutier,Mérigot,Thibert]

Contributions

1. equivalence par-regularity and reach
2. Hausdorff distance between ∂X and $\partial_h X$ for sets with positive reach
3. in 3D, localization of non-manifold places of $\partial_h X$
4. in nD, localization and quantification of non-injective places of $\xi_{\partial X}$ from $\partial_h X$ to ∂X
5. a multigrid convergent digital surface integration scheme in nD
   $\Rightarrow$convergent local area estimator given convergent normal estimator

Properties of Gauss digitized shapes, digital surface integration

1 Context and objectives

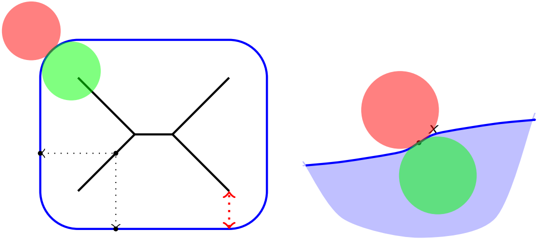
2 Properties of Gauss digitized sets

3 Manifoldness of digitized boundary

4 Injectiveness of projection

5 Digital surface integration

Par-regularity and positive reach

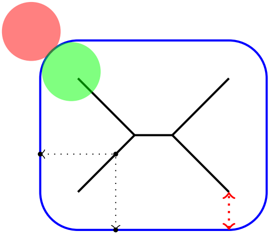


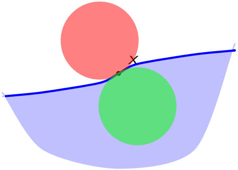
Lemma

Let X be a d-dimensional compact domain of $\mathbb{R}^d$. Then

$$\mathrm{reach}(\partial X) \geq R \quad \Leftrightarrow \quad \forall R' < R,\ X \text{ is } par(R')\text{-regular.}$$

Par-regularity and positive reach





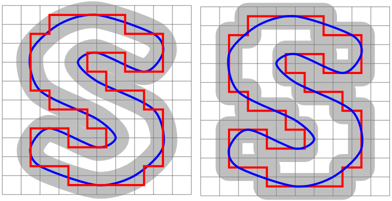
Lemma

Let X be a d-dimensional compact domain of $\mathbb{R}^d$. Then

$$\mathrm{reach}(\partial X) \geq R \quad \Leftrightarrow \quad \forall R' < R,\ X \text{ is } par(R')\text{-regular.}$$

If ∂X has positive reach greater than R, then, for $R' < R$ and $x \in \partial X$, there are inside and outside osculating balls of radius R' at x.

Hausdorff distance between continuous and digital boundary



Theorem

Let X be a compact domain of $\mathbb{R}^d$ such that the reach of ∂X is greater than R. Then, for any digitization step $0 < h < 2R/\sqrt{d}$, the Hausdorff distance between sets ∂X and $\partial_h X$ is less than $\sqrt{d}h/2$. More precisely:

$$\forall x \in \partial X, \exists y \in \partial_h X, \|x - y\| \leq \frac{\sqrt{d}}{2} h \quad (\text{with } \xi_{\partial X}(y) = x), \tag{2}$$

$$\forall y \in \partial_h X, \|y - \xi_{\partial X}(y)\| \leq \frac{\sqrt{d}}{2} h. \tag{3}$$

Remark that this bound is tight. The proof uses osculating balls and the fact that ∂X is at least C^1.

Properties of Gauss digitized shapes, digital surface integration

1 Context and objectives

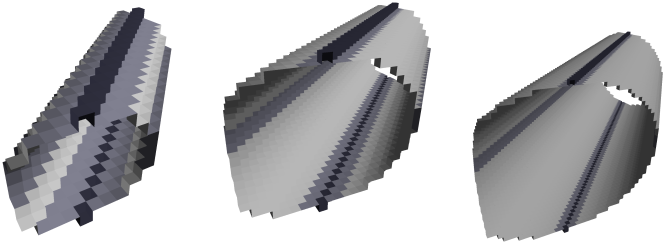
2 Properties of Gauss digitized sets

3 Manifoldness of digitized boundary

4 Injectiveness of projection

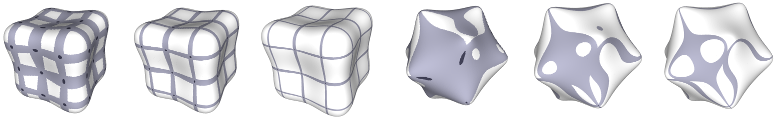
5 Digital surface integration

Non-manifold parts of digitized boundary



- In 3D, there are smooth shapes which are not digitized as manifolds whatever the gridstep. [Stelldinger et al. 2007]
- Problem related to cross configurations (i.e. critical [Latecki et al.])
- We locate and quantify non-manifold parts of digitized boundaries.

Manifoldness local sufficient condition



Theorem (Manifoldness sufficient condition in $\mathbb{R}^3$)

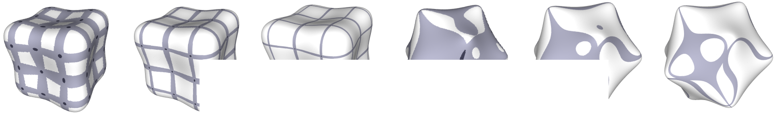
Let X be some compact domain of $\mathbb{R}^3$, with $\mathrm{reach}(\partial X)$ greater than some positive constant R and $h < 0.198R$. Let y be a point of $\partial_h X$.

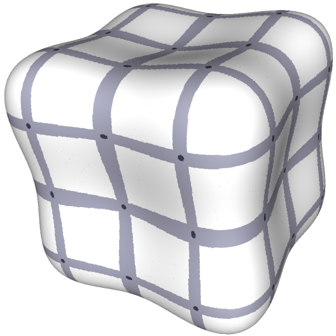
i) *If y does not belong to some 1-cell of $\partial_h X$ that intersect ∂X, then $\partial_h X$ is homeomorphic to a 2-disk around y.*

ii) *If y belongs to some 1-cell s of $\partial_h X$ such that $\partial X \cap s$ contains a point P and if the angle α_y between s and the normal to ∂X at P satisfies $\alpha_y \geq 1.260h/R$, then $\partial_h X$ is homeomorphic to a 2-disk around y.*

Only places where the normal is close to some axis may be non-manifold.

Manifoldness local sufficient condition





Theorem (Manifol

Let X be some com ... *ater than some positive constant R* ... *$_hX$.*

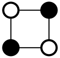
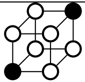
i) *If y does not b* ... *ect ∂X, then $\partial_h X$ is homeomorphic*

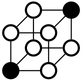
ii) *If y belongs to* ... *contains a point P and if the angl* ... *t P satisfies $\alpha_y \geq 1.260h/R$* ... *k around y.*

Only places where t ... non-manifold.

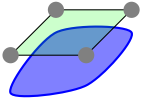
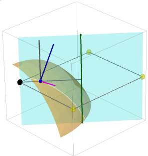
Main ingredients of the proof (I)

- Non-manifold parts of $\partial_h X$ only at "crossed" configurations of $\mathtt{G}_X(h)$:

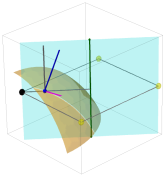
non-manifold edge 	non-manifold vertex 

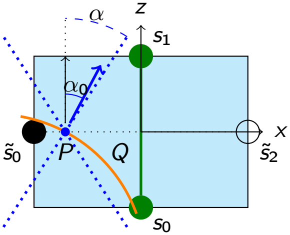
- no  for $h < R/2$ and par(R)-regularity, Theorem 13 of [Stelldinger et al. 2007]

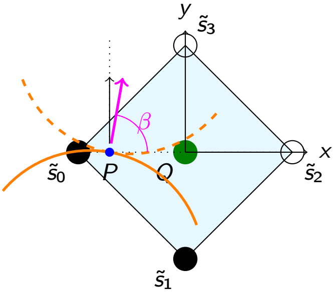
- Examine ∂X around each 4-tuple of $\mathbb{Z}^3$

1 $\partial X \cap$ dual cell $= \emptyset$	2,3 $\partial X \cap$ dual cell $\neq \emptyset$	
		
	2 $\partial X \cap$ 1-cell $= \emptyset$	3 $\partial X \cap$ 1-cell $\neq \emptyset$

Main ingredients of the proof (II)

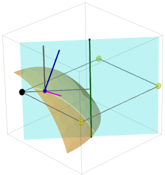


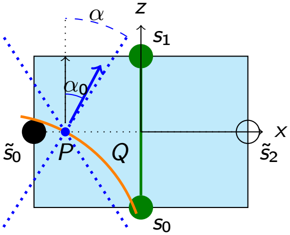


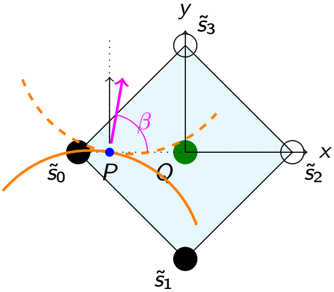


- 2 ∂X intersects dual cell and $\partial X \cap$ 1-cell $= \emptyset$
  0. equivalence reach / par-regularity implies inside/outside osculating balls at P

Main ingredients of the proof (II)

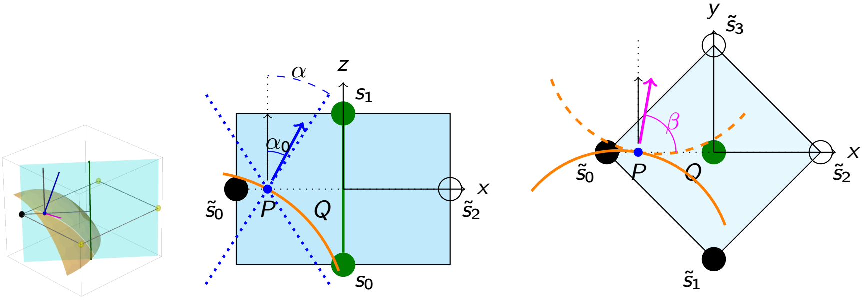






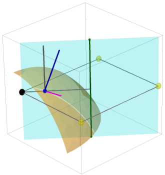
- 2 ∂X intersects dual cell and $\partial X \cap$ 1-cell $= \emptyset$
  0. equivalence reach / par-regularity implies inside/outside osculating balls at P
  1. angle α at P cannot be too small when $h \to 0$

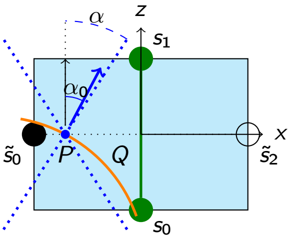
Main ingredients of the proof (II)

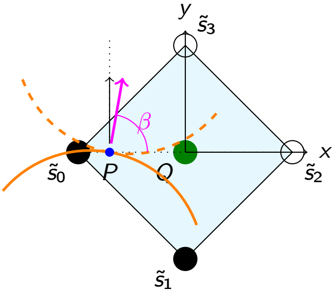


- 2 ∂X intersects dual cell and $\partial X \cap$ 1-cell $= \emptyset$
  0. equivalence reach / par-regularity implies inside/outside osculating balls at P
  1. angle α at P cannot be too small when $h \to 0$
  2. residual radius of osculating ball is $R/sin\alpha$

Main ingredients of the proof (II)

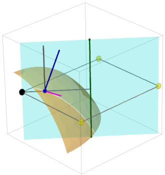


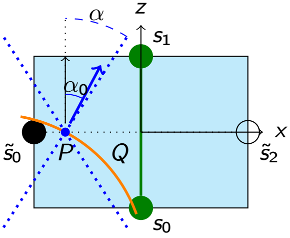


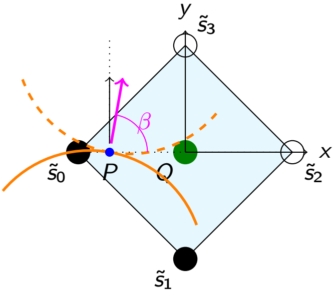


- 2 ∂X intersects dual cell and $\partial X \cap$ 1-cell $= \emptyset$
  0. equivalence reach / par-regularity implies inside/outside osculating balls at P
  1. angle α at P cannot be too small when $h \to 0$
  2. residual radius of osculating ball is $R/sin\alpha$
  3. either $\tilde{s}_1 \in \mathtt{G}_h(X)$ or $\tilde{s}_2 \notin \mathtt{G}_h(X)$ for $\frac{h}{\sin\alpha} < \frac{\sqrt{26}}{13}$.

Main ingredients of the proof (II)

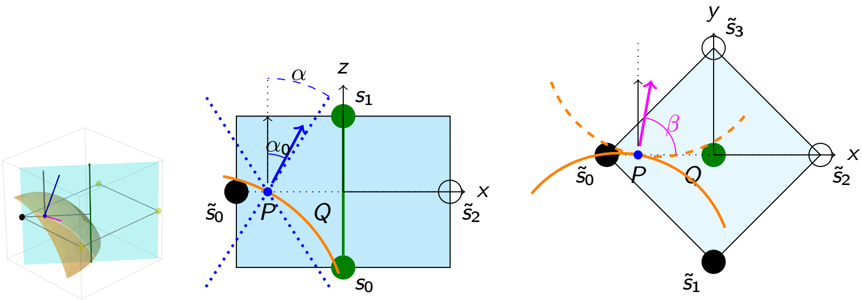






- 2 ∂X intersects dual cell and $\partial X \cap$ 1-cell $= \emptyset$
  0. equivalence reach / par-regularity implies inside/outside osculating balls at P
  1. angle α at P cannot be too small when $h \to 0$
  2. residual radius of osculating ball is $R/sin\alpha$
  3. either $\tilde{s}_1 \in \mathtt{G}_h(X)$ or $\tilde{s}_2 \notin \mathtt{G}_h(X)$ for $\frac{h}{\sin \alpha} < \frac{\sqrt{26}}{13}$.
  4. balance 1 and 3 to get $h < 0.198R \Rightarrow$ non-crossed.

Main ingredients of the proof (II)



- 2 ∂X intersects dual cell and $\partial X \cap$ 1-cell $= \emptyset$
- 3 ∂X intersects dual cell and $\partial X \cap$ 1-cell $\neq \emptyset$

Properties of Gauss digitized shapes, digital surface integration

1 Context and objectives

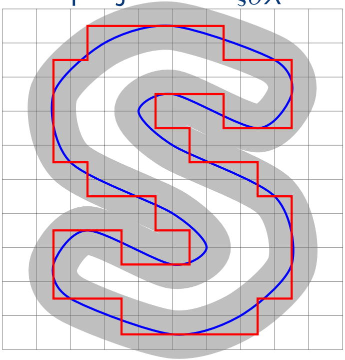
2 Properties of Gauss digitized sets

3 Manifoldness of digitized boundary

4 Injectiveness of projection

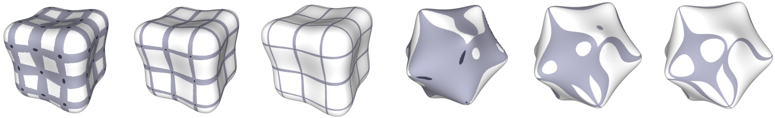
5 Digital surface integration

Non-injective part of projection $\xi_{\partial X}$



- projection $\xi_{\partial X}$ defines a natural relation between $\partial_h X$ and ∂X.
- this projection is not everywhere injective
- we wish to know where and to quantify this part
- we will thus be able to prove the convergence of digital surface integration

Size of non-injective part of $\xi_{\partial X}$



The set $\mathrm{mult}(\partial X)$ defines the part of ∂X where the projection is not injective.

$$\mathrm{mult}(\partial X) := \{x \in \partial X, \text{s.t. } \exists y_1, y_2 \in \partial_h X, y_1 \neq y_2,\ \xi(y_1) = \xi(y_2) = x\}.$$

Theorem

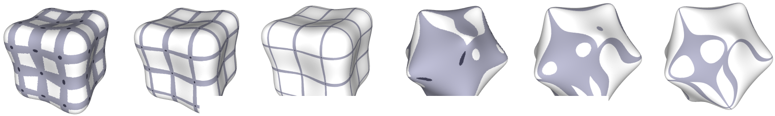
If $h \leq R/\sqrt{d}$, then one has

$$\mathrm{Area}(\mathrm{mult}(\partial X)) \leq K_1(h)\ \mathrm{Area}(\partial X)\ h,$$

where

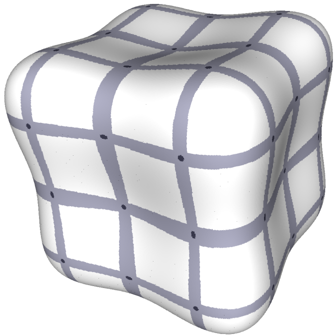
$$K_1(h) = \frac{8d^2}{R} + O(h) \leq \frac{d^2\ 4^{d+1}}{R}.$$

Size of non-injective part of $\xi_{\partial X}$



The set $\mathrm{mult}(\partial X)$ c ion is not injective.

$$\mathrm{mult}(\partial X) := \{x \quad) = \xi(y_2) = x\}.$$

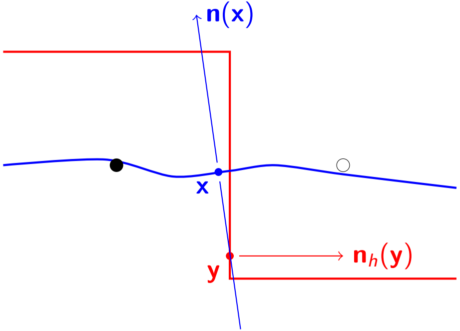


Theorem

If $h \leq R/\sqrt{d}$, the

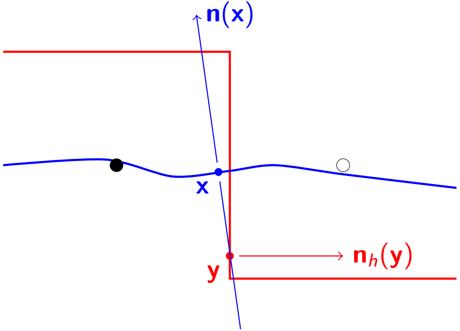
where

Some elements of the proof (I)



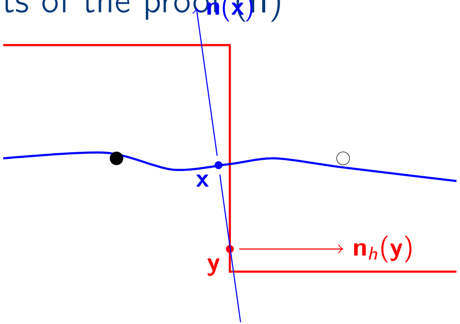
- since $h \leq R/\sqrt{d}$, former theorem implies $d_H(\partial X, \partial_h X) < \sqrt{d}h/2$, and restriction of ξ to $\partial_h X$ is surjective
- let $\mathrm{mult}(\partial_h X) := \xi^{-1}(\mathrm{mult}(\partial X))$
- then $\xi : \partial_h X \setminus \mathrm{mult}(\partial_h X) \to \partial X \setminus \mathrm{mult}(\partial X)$ is one-to-one.

Some elements of the proof (I)



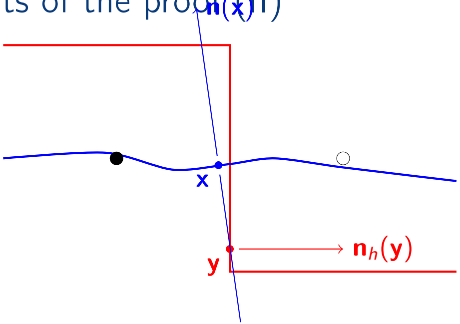
- since $h \leq R/\sqrt{d}$, former theorem implies $d_H(\partial X, \partial_h X) < \sqrt{d}h/2$, and restriction of ξ to $\partial_h X$ is surjective
- let $\mathrm{mult}(\partial_h X) := \xi^{-1}(\mathrm{mult}(\partial X))$
- then $\xi : \partial_h X \setminus \mathrm{mult}(\partial_h X) \to \partial X \setminus \mathrm{mult}(\partial X)$ is one-to-one.
- Difficulty: $\mathrm{mult}(\partial X)$ will be small but $\mathrm{mult}(\partial_h X)$ is maybe not negligible.

Some elements of the proof (II)



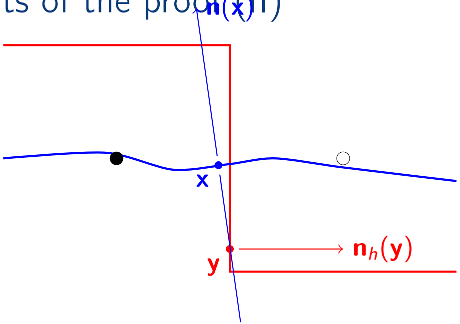
1. Scalar product between normals of $\partial_h X$ and $\partial X \geq -\frac{\sqrt{3d}}{R} h$.
   Use the fact that **n** are Lipschitz over ∂X (explicit [Federer59]).

Some elements of the proof (II)



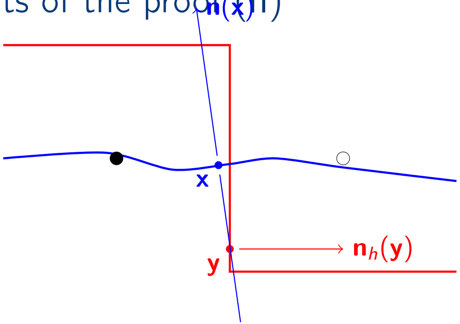
1. Scalar product between normals of $\partial_h X$ and $\partial X \geq -\frac{\sqrt{3d}}{R} h$.
   Use the fact that $\mathbf{n}$ are Lipschitz over ∂X (explicit [Federer59]).
2. Then $\mathrm{mult}(\partial X) \subset \xi(P(h))$, with $P(h) := \{\mathbf{y} \in \partial_h X,\ \mathbf{n}(\xi(\mathbf{y})) \cdot \mathbf{n}_h(\mathbf{y}) \leq 0\}$
   Observe the intersections of segment $\mathbf{n}(\mathbf{x})$ with faces of $\partial_h X$.

Some elements of the proof (II)



1. Scalar product between normals of $\partial_h X$ and $\partial X \geq -\frac{\sqrt{3d}}{R}h$.
   Use the fact that $\mathbf{n}$ are Lipschitz over ∂X (explicit [Federer59]).
2. Then $\mathrm{mult}(\partial X) \subset \xi(P(h))$, with $P(h) := \{\mathbf{y} \in \partial_h X,\ \mathbf{n}(\xi(\mathbf{y})) \cdot \mathbf{n}_h(\mathbf{y}) \leq 0\}$
   Observe the intersections of segment $\mathbf{n}(\mathbf{x})$ with faces of $\partial_h X$.
3. The jacobian of ξ at $\mathbf{y}$ is $\approx |\mathbf{n}(\xi(\mathbf{y})) \cdot \mathbf{n}_h(\mathbf{y})|$, hence the jacobian of its restriction to $P(h)$ is in $O(h)$.
   Use geometric measure theory on $D\xi$ and $\|\mathbf{y} - \xi(\mathbf{y})\| < \sqrt{d}h/2$, [Morvan08]

Some elements of the proof (II)



1. Scalar product between normals of $\partial_h X$ and $\partial X \geq -\frac{\sqrt{3d}}{R}h$.
   Use the fact that $\mathbf{n}$ are Lipschitz over ∂X (explicit [Federer59]).
2. Then $\mathrm{mult}(\partial X) \subset \xi(P(h))$, with $P(h) := \{\mathbf{y} \in \partial_h X,\ \mathbf{n}(\xi(\mathbf{y})) \cdot \mathbf{n}_h(\mathbf{y}) \leq 0\}$
   Observe the intersections of segment $\mathbf{n}(\mathbf{x})$ with faces of $\partial_h X$.
3. The jacobian of ξ at $\mathbf{y}$ is $\approx |\mathbf{n}(\xi(\mathbf{y})) \cdot \mathbf{n}_h(\mathbf{y})|$, hence the jacobian of its restriction to $P(h)$ is in $O(h)$.
   Use geometric measure theory on $D\xi$ and $\|\mathbf{y} - \xi(\mathbf{y})\| < \sqrt{d}h/2$, [Morvan08]
4. We conclude that $\mathrm{Area}(\mathrm{mult}(\partial X))$ is in $O(h)$.

Properties of Gauss digitized shapes, digital surface integration

1 Context and objectives

2 Properties of Gauss digitized sets

3 Manifoldness of digitized boundary

4 Injectiveness of projection

5 Digital surface integration

Digital surface integral

Definition

Let $Z \subset (h\mathbb{Z})^d$ be a digital set. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an integrable function and $\hat{\mathbf{n}}$ be a digital normal estimator. We define the *digital surface integral* by

$$\mathrm{DI}_h(f, Z, \hat{\mathbf{n}}) := \sum_{d-1\text{-cell}\, c \in \partial[Z]_h} h^{d-1} f(\dot{c}) |\hat{\mathbf{n}}(\dot{c}) \cdot \mathbf{n}(\dot{c})|,$$

where $\dot{c}$ is the centroid of the $(d-1)$-cell c and $\mathbf{n}(\dot{c})$ is its trivial normal as a point on the h-boundary $\partial_h X$.

Digital surface integral

Definition

Let $Z \subset (h\mathbb{Z})^d$ be a digital set. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an integrable function and $\hat{\mathbf{n}}$ be a digital normal estimator. We define the *digital surface integral* by

$$\mathrm{DI}_h(f, Z, \hat{\mathbf{n}}) := \sum_{d-1\text{-cell} c \in \partial[Z]_h} h^{d-1} f(\dot{c}) |\hat{\mathbf{n}}(\dot{c}) \cdot \mathbf{n}(\dot{c})|,$$

where $\dot{c}$ is the centroid of the $(d-1)$-cell c and $\mathbf{n}(\dot{c})$ is its trivial normal as a point on the h-boundary $\partial_h X$.

Theorem

Let X be a compact domain where ∂X has reach greater than R. For $h \leq \frac{R}{\sqrt{d}}$, the digital integral is multigrid convergent toward the integral over ∂X.

$$\left| \int_{\partial X} f(x)dx - \mathrm{DI}_h(f, \mathtt{G}_h(X), \hat{\mathbf{n}}) \right| \leq \mathrm{Area}(\partial X) \; \|f\|_{\mathrm{BL}} \; \Big(O(h) + O(\|\hat{\mathbf{n}} - \mathbf{n}\|_{\mathrm{est}}) \Big).$$

Steps of the proof

1. First $\int_{\partial X} f(x)dx = \int_{\partial X \setminus \mathrm{mult}(\partial X)} f(x)dx + K_1(h)\mathrm{Area}(\partial X)\|f\|_\infty h$.
   (size of non injective part).

Steps of the proof

1. First $\int_{\partial X} f(x)dx = \int_{\partial X \setminus \mathrm{mult}(\partial X)} f(x)dx + K_1(h)\mathrm{Area}(\partial X)\|f\|_\infty h$.
   (size of non injective part).
2. Then, $\int_{\partial X \setminus \mathrm{mult}(\partial X)} f(x)dx = \int_{\partial_h X \setminus \mathrm{mult}(\partial_h X)} f(\xi(y))\xi(y)dy$.
   (diffeomorphism of ξ + change of variable formula)

Steps of the proof

1. First $\int_{\partial X} f(x)dx = \int_{\partial X \setminus \mathrm{mult}(\partial X)} f(x)dx + K_1(h)\mathrm{Area}(\partial X)\|f\|_\infty h$.
   (size of non injective part).
2. Then, $\int_{\partial X \setminus \mathrm{mult}(\partial X)} f(x)dx = \int_{\partial_h X \setminus \mathrm{mult}(\partial_h X)} f(\xi(y))\xi(y)dy$.
   (diffeomorphism of ξ + change of variable formula)
3. $\left| \int_{\partial_h X \setminus \mathrm{mult}(\partial_h X)} f(\xi(y))\xi(y)dy - \int_{\partial_h X} f(\xi(y))\xi(y)dy \right| \leq$
   $\mathrm{Area}(\partial X)\ \mu\ \|f\|_\infty O(h)$
   (multiplicity ae bounded by $\mu := d\lfloor\sqrt{d}+1\rfloor$ and coarea formula)

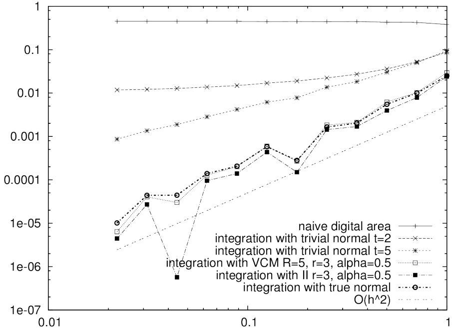
Steps of the proof

1. First $\int_{\partial X} f(x)dx = \int_{\partial X \setminus \mathrm{mult}(\partial X)} f(x)dx + K_1(h)\mathrm{Area}(\partial X)\|f\|_\infty h$.
   (size of non injective part).
2. Then, $\int_{\partial X \setminus \mathrm{mult}(\partial X)} f(x)dx = \int_{\partial_h X \setminus \mathrm{mult}(\partial_h X)} f(\xi(y))\xi(y)dy$.
   (diffeomorphism of ξ + change of variable formula)
3. $\left| \int_{\partial_h X \setminus \mathrm{mult}(\partial_h X)} f(\xi(y))\xi(y)dy - \int_{\partial_h X} f(\xi(y))\xi(y)dy \right| \leq$
   $\mathrm{Area}(\partial X)\ \mu\ \|f\|_\infty O(h)$
   (multiplicity ae bounded by $\mu := d\lfloor\sqrt{d}+1\rfloor$ and coarea formula)
4. $\int_{\partial_h X} f(\xi(y))\xi(y)dy =$
   $\int_{\partial_h X} f(\xi(y))|\mathbf{n}(\xi(y)) \cdot \mathbf{n}_h(y)|dy + \|f\|_\infty \mathrm{Area}(\partial X) O(h)$.
   (Jacobian property and upper bound on $\partial_h X$ area)

Steps of the proof

1. First $\int_{\partial X} f(x)dx = \int_{\partial X \setminus \mathrm{mult}(\partial X)} f(x)dx + K_1(h)\mathrm{Area}(\partial X)\|f\|_\infty h$.
   (size of non injective part).
2. Then, $\int_{\partial X \setminus \mathrm{mult}(\partial X)} f(x)dx = \int_{\partial_h X \setminus \mathrm{mult}(\partial_h X)} f(\xi(y))\xi(y)dy$.
   (diffeomorphism of ξ + change of variable formula)
3. $\left| \int_{\partial_h X \setminus \mathrm{mult}(\partial_h X)} f(\xi(y))\xi(y)dy - \int_{\partial_h X} f(\xi(y))\xi(y)dy \right| \leq$
   $\mathrm{Area}(\partial X)\ \mu\ \|f\|_\infty O(h)$
   (multiplicity ae bounded by $\mu := d\lfloor\sqrt{d}+1\rfloor$ and coarea formula)
4. $\int_{\partial_h X} f(\xi(y))\xi(y)dy =$
   $\int_{\partial_h X} f(\xi(y))|\mathbf{n}(\xi(y)) \cdot \mathbf{n}_h(y)|dy + \|f\|_\infty \mathrm{Area}(\partial X)O(h)$.
   (Jacobian property and upper bound on $\partial_h X$ area)
5. $\left| \int_{\partial_h X} f(\xi(y))|\mathbf{n}(\xi(y)) \cdot \mathbf{n}_h(y)|dy - \mathrm{DI}_h(f, \mathtt{G}_h(X), \hat{\mathbf{n}}) \right| \leq$
   $\mathrm{Area}(\partial X)\Big(\mathrm{Lip}(f)O(h) + \|f\|_\infty O(\|\hat{\mathbf{n}} - \mathbf{n}\|_{\mathrm{est}})\Big)$.
   (sum cell by cell plus error between $\mathbf{n}(\xi(\mathbf{y}))$ and $\hat{\mathbf{n}}(c)$)

Experimental evaluation



Area estimation error of the digital surface integral with several digital normal estimators. The shape of interest is 3D ellipsoid of half-axes 10, 10 and 5, for which the area has an analytical formula giving $A \approx 867.188270334505$. The abscissa is the gridstep h at which the ellipsoid is sampled by Gauss digitization. For each normal estimator, the digital surface integral $\hat{A}$ is computed with $f = 1$, and the relative area estimation error $\frac{|\hat{A}-A|}{A}$ is displayed in logscale.

F. Chazal and A. Lieutier.
Smooth manifold reconstruction from noisy and non-uniform approximation with guarantees.
Computational Geometry, 40(2):156–170, 2008.

D. Coeurjolly, F. Flin, O. Teytaud, and L. Tougne.
Multigrid convergence and surface area estimation.
In *Geometry, morphology, and computational imaging*, pages 101–119. Springer, 2003.

D. Coeurjolly, J.-O. Lachaud, and J. Levallois.
Integral based curvature estimators in digital geometry.
In *Discrete Geometry for Computer Imagery*, number 7749 in LNCS, pages 215–227. Springer, 2013.

D. Coeurjolly, J.-O. Lachaud, and J. Levallois.
Multigrid convergent principal curvature estimators in digital geometry.
Computer Vision and Image Understanding, (0):–, 2014.
In Press, Corrected Proof, Available online 9 May 2014.

L. Cuel, J.-O. Lachaud, and B. Thibert.
Voronoi-based geometry estimator for 3d digital surfaces.
In *Proc. Discrete Geometry for Computer Imagery (DGCI'2014), Sienna, Italy*, Lecture Notes in Computer Science. Springer, 2014.

F. de Vieilleville, J.-O. Lachaud, and F. Feschet.
Maximal digital straight segments and convergence of discrete geometric estimators.
Journal of Mathematical Image and Vision, 27(2):471–502, February 2007.

H.-A. Esbelin and R. Malgouyres.
Convergence of binomial-based derivative estimation for c2-noisy discretized curves.
In *Proc. 15th DGCI*, volume 5810 of *LNCS*, pages 57–66. 2009.

H.-A. Esbelin, R. Malgouyres, and C. Cartade.
Convergence of binomial-based derivative estimation for c^2 noisy discretized curves.
Theoretical Computer Science, 412(36):4805–4813, 2011.

H. Federer.
Curvature measures.
Trans. Amer. Math. Soc, 93(3):418–491, 1959.

H. Federer
Geometric measure theory.
Springer, 1969.

M. N. Huxley.
Exponential sums and lattice points.
Proc. London Math. Soc., 60:471–502, 1990.

R. Klette and A. Rosenfeld.
Digital Geometry: Geometric Methods for Digital Picture Analysis.
Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 2004.

R. Klette and J. Žunić.
Multigrid convergence of calculated features in image analysis.
Journal of Mathematical Imaging and Vision, 13(3):173–191, 2000.

J.-O. Lachaud, A. Vialard, and F. de Vieilleville.
Fast, accurate and convergent tangent estimation on digital contours.
Image and Vision Computing, 25(10):1572 – 1587, 2007.

L. J. Latecki.
3D well-composed pictures.
Graphical Models and Image Processing, 59(3):164–172, May 1997.

L. J. Latecki, C. Conrad, and A. Gross.
Preserving topology by a digitization process.
Journal of Mathematical Imaging and Vision, 8(2):131–159, mar 1998.

A. Lenoir, R. Malgouyres, and M. Revenu.
Fast computation of the normal vector field of the surface of a 3-d discrete object.
In *Discrete Geometry for Computer Imagery*, pages 101–112. Springer, 1996.

Y.-S. Liu, J. Yi, H. Zhang, G.-Q. Zheng, and J.-C. Paul.
Surface area estimation of digitized 3d objects using quasi-monte carlo methods.
Pattern Recognition, 43(11):3900 – 3909, 2010.

P. Niyogi, S. Smale, and S. Weinberger.
Finding the homology of submanifolds with high confidence from random samples.
Discrete & Computational Geometry, 39(1-3):419–441, 2008.

T. Pavlidis.
Algorithms for graphics and image processing.
Computer science press, 1982.

L. Provot and Y. Gérard.
Estimation of the derivatives of a digital function with a convergent bounded error.
In *Discrete Geometry for Computer Imagery*, pages 284–295. Springer, 2011.

C. Ronse and M. Tajine.
Discretization in hausdorff space.
Journal of Mathematical Imaging and Vision, 12(3):219–242, 2000.

J. Serra.
Image analysis and mathematical morphology.
Academic Press, 1982.

P. Stelldinger and U. Köthe.
Towards a general sampling theory for shape preservation.
Image and Vision Computing, 23(2):237–248, 2005.

P. Stelldinger, L. J. Latecki, and M. Siqueira.
Topological equivalence between a 3d object and the reconstruction of its digital image.
IEEE Transactions on Pattern Analysis and Machine Intelligence, 29(1):126–140, 2007.

M. Tajine and C. Ronse.
Topological properties of hausdorff discretization, and comparison to other discretization schemes.
Theoretical Computer Science, 283(1):243–268, 2002.

H. Weyl.
On the volume of tubes,
American Journal of Mathematics, 61(2):461–472, 1939.