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# Characterization of bijective discretized rotations by Gaussian integers

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**Résumé**
*Une rotation discrète est la composition d'une rotation euclidienne et d'une opération d'arrondi. Bien sûr, toutes les rotations discrètes ne sont pas bijectives : par exemple, deux points distincts peuvent avoir la même image pour une rotation discrète donnée. Néanmoins, pour un certain ensemble d'angles, les rotations discrètes sont bijectives. Dans la grille carrée régulière, les rotations discrètes bijectives ont été complètement caractérisées par Nouvel et Rémila (IWCIA'2005). Nous donnons une preuve qui utilise les propriétés arithmétiques des entiers de Gauss.*

*A discretized rotation is the composition of an Euclidean rotation with a rounding operation. It is well known that not all discretized rotations are bijective : e.g. two distinct points may have the same image by a given discretized rotation. Nevertheless, for a certain subset of rotation angles, the discretized rotations are bijective. In the regular square grid, the bijective discretized rotations have been fully characterized by Nouvel and Rémila (IWCIA'2005). We provide a simple proof that uses the arithmetical properties of Gaussian integers.*

**Mots clé :** Gaussian integers, bijective discretized rotations.

## 1. Introduction

A discretized rotation is the composition of an Euclidean rotation with a rounding operation to the closest grid point. It is well known that not all discretized rotations are bijective : after a discretized rotation, two distinct points may have the same image (see Fig. 1.b) or the image of all points may not partition the whole plane (the reader may look for the *holes* in Fig. 1.b). Nevertheless, for a certain subset of rotation angles, the discretized rotations are bijective (an example of such discretized rotations is shown Fig. 1.a). In the regular square grid, many authors have discussed about conditions on the angle to have bijective discretized rotations. In [NR05], Nouvel and Rémila have fully characterized bijective discretized rotations (necessary and sufficient conditions on rotation angles). We give in this paper a different proof that uses simple arithmetical properties of Gaussian integers.

In section 2, we recall the crucial properties of Gaussian integers and we give a geometrical interpretation of main arithmetical operations involving Gaussian integers. In section 3, we define a discretized rotation and characterize a certain set of rotation angles by the so-called *twin Pythagorean triples*. Finally, in section 4, we show theorem 1, which provides a necessary and sufficient condition for rotation angles to lead to bijective discretized rotations.

## 2. Gaussian integers

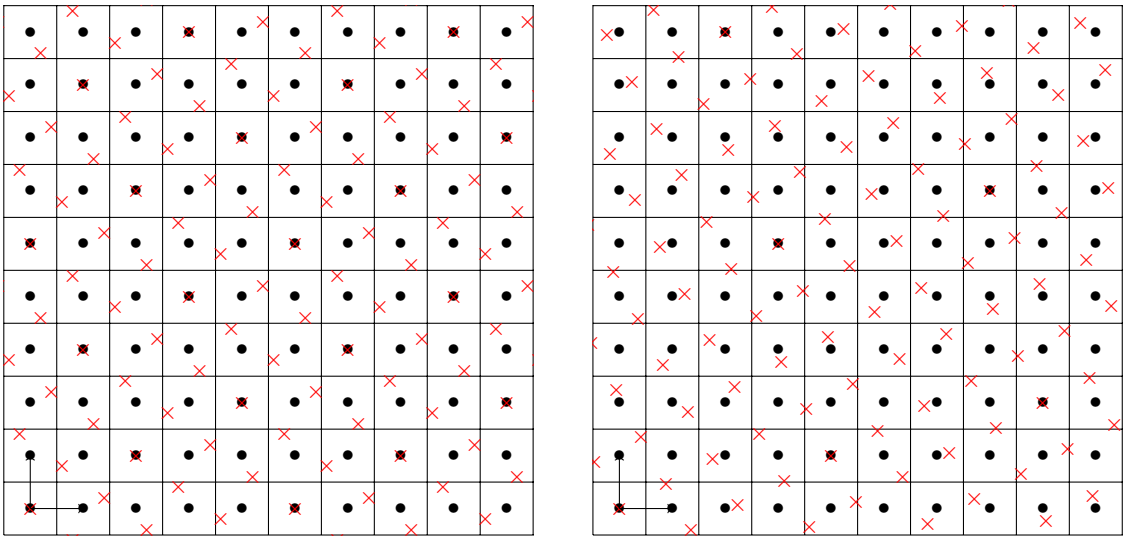
The Gaussian integers are the set $\mathbb{Z}[i] := \{u+vi \mid u,v \in \mathbb{Z}\}$, where $i^2 = -1$. Within the complex plane $\mathbb{C}$, they constitute the 2-dimensional integer lattice $\mathbb{Z}^2$.

### 2.1. Main properties

As discussed in [HW79][pp. 182–187], Gaussian integers look like usual (or *rational*) integers of $\mathbb{Z}$. Indeed, the notions of Euclidean division, prime, greatest common divisor are defined. Moreover, every Gaussian integer has a unique factorization into primes (up to order and unit multiples).

More precisely, let $\kappa, \kappa_j$ be nonzero integers from $\mathbb{Z}[i]$.

— The *norm* of $\kappa = u+vi$, defined by $N\kappa := \kappa\bar{\kappa} = u^2+v^2$ is multiplicative, i.e. $N\kappa\kappa_1 = N\kappa N\kappa_1$.
— The units of $\mathbb{Z}[i]$ are the integers of norm 1, i.e. the set $\{\pm 1, \pm i\}$.
— Since there are several units, the product of $\kappa = u+vi$ by any number of units, i.e. the four integers $\pm u \pm vi$, are the *associates* of $\kappa$ ;



(a) (b)

**Figure 1:** *In (a), the rotation by angle* $\theta_a$ *s.t.* $\tan(\theta_a) = 4/3$ *leads to a bijective discretized rotation because each digitization cell (black squares around black dots) contains one and only one rotated point (red dots). In (b), the rotation angle* $\theta_b$ *s.t.* $\tan(\theta_b) = 8/15$ *does not lead to a bijective discretized rotation : some digitization cells contain zero (*holes*) or two points.*

— $\kappa$ is *divisible* by $\kappa_1$ iff there exists $\kappa_2$ such that $\kappa = \kappa_1\kappa_2$ ;
— A *prime* is an integer, neither zero nor a unit, divisible only by numbers associated to itself or 1 ;
— Any $\kappa$ can be obtained as a product of primes (unique up to order and unit multiples) : $\kappa = \pi_1\pi_2\dots\pi_n$.
— The greatest common divisor $gcd(\kappa,\kappa_1) = \kappa_2$ is defined such that (i) $\kappa_2$ divides both $\kappa$ and $\kappa_1$ and (ii) every common divisor of $\kappa$ and $\kappa_1$ divides $\kappa_2$.

For a complete overview, please refer to [HW79]. We focus now on the geometrical interpretation of Gaussian integers.

### 2.2. Geometrical interpretation

Gaussian integers are complex numbers. The image of a Gaussian integer $\kappa = u + vi$ is the point $(u,v)$ of the integer lattice $\mathbb{Z}^2$.

Let us first observe that :
— an addition by $\kappa$ maps $\mathbb{Z}^2$ to $\mathbb{Z}^2 + (u,v)$ (translation).
— a multiplication by $\kappa$ maps $\mathbb{Z}^2$ to $\mathbb{Z}(u,v) + \mathbb{Z}(-v,u)$ (rotation by angle $\theta$ such that $\tan(\theta) = v/u$ and scaling by $\sqrt{N\kappa}$; see Fig. 2).

## 3. Discretized rotations

Given a Gaussian integer $\alpha$, the Euclidean rotation is the map defined as follows :

$$\begin{aligned} &r_\alpha : \mathbb{Z}[i] \to \mathbb{C} \\ &\forall \kappa \in \mathbb{Z}[i],\ r_\alpha(\kappa) = \tfrac{\kappa\cdot\alpha}{\sqrt{N\alpha}}. \end{aligned} \tag{1}$$

Moreover, we focus on *Pythagorean* rotation angles, i.e. such that $\sqrt{N\alpha} = c \in \mathbb{Z}$.

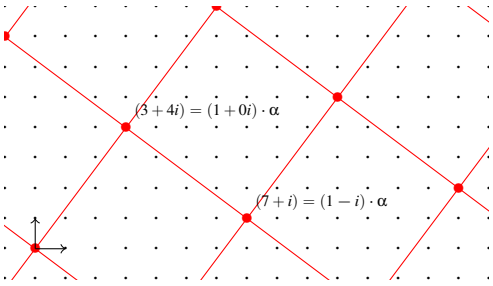



**Figure 2:** *A multiplication by* $\alpha := (3+4i)$ *results in a rotation of angle* $\theta$ *s.t.* $\tan(\theta) = 4/3$ *and a scaling by* 5. *Moreover, the image of any Gaussian integer that is a multiple of* $\alpha$ *(red dots) is a point of the lattice* $\mathbb{Z}(3,4) + \mathbb{Z}(-4,3)$ *(red grid).*

### 3.1. Pythagorean triples

Pythagorean triples are triples $(a,b,c)$ of strictly positive integers such that $a^2 + b^2 = c^2$. Setting $\alpha := a + bi$, Pythagorean triples provide solutions to the equation : $N\alpha = c^2$. *Primitive* Pythagorean triples are such that $\gcd(a,b,c) = 1$.

It is well known [HW79][p. 190] that for any primitive Pythagorean triple, there exists a unique pair $(p,q)$ of positive integers such that $0 < q < p$, $\gcd(p,q) = 1$, $p - q$ is odd and

$$\begin{aligned} a &= p^2 - q^2, \\ b &= 2pq, \\ c &= p^2 + q^2. \end{aligned}$$

A specific family of Pythagorean triples is the so-called

*twin Pythagorean triples* or $(k+1,k)$-family, where $p=k+1$ and $q=k$.

Setting $\gamma := p+qi$, we have on the one hand

$$\alpha = \gamma \cdot \gamma, \tag{2}$$

and on the other hand

$$c = \gamma \cdot \bar{\gamma}. \tag{3}$$

First, $\gamma$ is neither divisible by a rational integer because $p$ and $q$ are coprime, nor divisible by a unit because $p-q$ is odd. Second, we have $gcd(\gamma,\bar{\gamma}) = 1$ because if we denote $\gamma = \pi_1\pi_2\dots\pi_n$, we have $\bar{\gamma} = \bar{\pi_1}\bar{\pi_2}\dots\bar{\pi_n}$ and thus no factor of $\gamma$ is also a factor of $\bar{\gamma}$. As a consequence, since $\gamma$ divides both $\alpha$ and $c$, we have

$$gcd(\alpha, c) = \gamma. \tag{4}$$

### 3.2. Discretization

For any $\kappa = u+iv$, let the *discretization cell* of $\kappa$ be defined as follows :

$$D(\kappa) = \left\{ z = x+iy \in \mathbb{C} \mid \begin{cases} u-1/2 \leq (u+x) < u+1/2 \\ v-1/2 \leq (v+y) < v+1/2 \end{cases} \right\}.$$

Geometrically, the discretization cell is an isothetic unit square around an integer point (see Fig. 1).

The *rounding function* is now defined as a function $\mathbb{C} \to \mathbb{Z}[i]$ such that $\forall z = x+iy \in \mathbb{C}$, $[z]$ is the unique Gaussian integer such that $z \in D([z])$.

Let us denote by $[r_\alpha]$ the composition of the Euclidean rotation $r_\alpha$ and of the rounding function $[.]$, i.e.

$$\begin{aligned} &[r_\alpha] : \mathbb{Z}[i] \to \mathbb{Z}[i] \\ &\forall \kappa \in \mathbb{Z}[i],\ [r_\alpha](\kappa) = \left[ \frac{\kappa \cdot \alpha}{\sqrt{N\alpha}} \right]. \end{aligned} \tag{5}$$

The goal of the next section is to prove the following theorem :

**Theorem 1** The discretized rotation $[r_\alpha]$ is bijective iff $\alpha = (k+1)+ki$, $k \in \mathbb{Z}^+$.

This result is equivalent to Nouvel and Remila's one [NR05]. However, in the following section, we prove this theorem using arithmetical and geometrical properties of Gaussian integers.

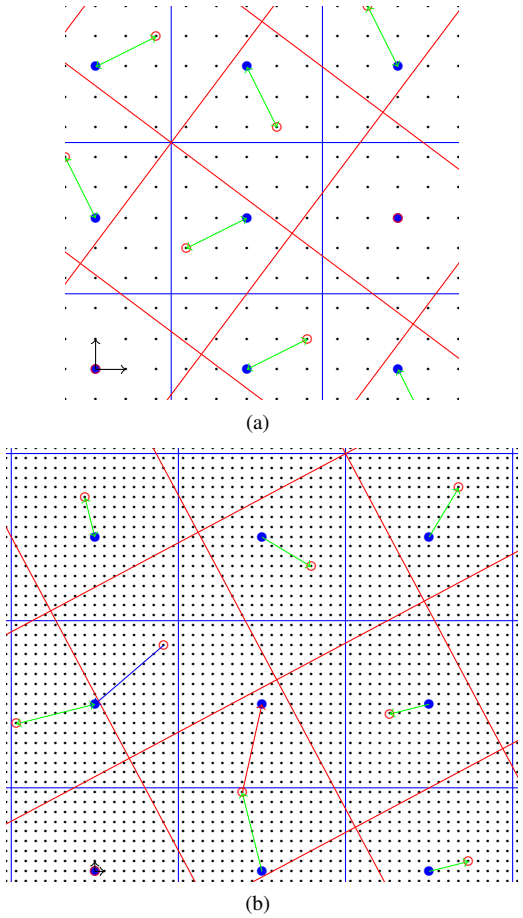
## 4. Main result

Until now, we divide $\kappa \cdot \alpha$ by $\sqrt{N\alpha} = c$ and then we consider the result with respect to the discretization cells of the integer lattice $\mathbb{Z}^2$ (Eq. 5). In this section, we do not divide $\kappa \cdot \alpha$ by $c$, but we consider the result with respect to the discretization cells of the scaled lattice $c\mathbb{Z}^2$ (see Fig. 3). In this framework, we introduce the map $s_{\alpha,c}$ defined as follows :

$$\begin{aligned} &s_{\alpha,c} : \mathbb{Z}[i] \times \mathbb{Z}[i] \to \mathbb{Z}[i] \\ &\forall (\kappa,\lambda) \in \mathbb{Z}[i] \times \mathbb{Z}[i],\ s_{\alpha,c}(\kappa,\lambda) := \kappa \cdot \alpha - \lambda \cdot c. \end{aligned} \tag{6}$$

### 4.1. Approach

The idea is to compare the points of the lattice $\mathbb{Z}(a,b) + \mathbb{Z}(-b,a)$ (i.e. the images of $\kappa \cdot \alpha$, $\kappa \in \mathbb{Z}[i]$, in red, Fig. 3) to the lattice $c\mathbb{Z}$ (i.e. the images of $\lambda \cdot c$, $\lambda \in \mathbb{Z}[i]$, in blue Fig. 3). However, instead of comparing any pair $(\kappa \cdot \alpha, \lambda \cdot c)$, $\kappa, \lambda \in \mathbb{Z}[i]$, we focus either on pairs such that $\kappa \cdot \alpha \in cD(\lambda)$, or on pairs such that $\lambda \cdot c \in \alpha D(\kappa)$. Such pairs are depicted with arrows in Fig. 3.



(a) (b)

**Figure 3:** *In (a) (resp. (b)), discretization cells of the lattice $5\mathbb{Z}^2$ (resp. $17\mathbb{Z}^2$) are depicted in blue, whereas discretization cells of the lattice $\mathbb{Z}(3,4)+\mathbb{Z}(-4,3)$ (resp. $\mathbb{Z}(15,8)+\mathbb{Z}(-8,15)$) are depicted in red. In both subfigures, blue (resp. red) arrows associate every red (resp. blue) point to the center of the blue (resp. red) discretization cell it belongs to. We use green for arrows that must be both blue and red.*

Indeed, for all $\kappa \in \mathbb{Z}[i]$, the proposition $\lambda = [r_\alpha(\kappa)]$, i.e. $\lambda \in D(\kappa \cdot \alpha / c)$, is equivalent to the proposition $\lambda \cdot c \in cD(\kappa \cdot \alpha)$, which is equivalent to the proposition

$$s_{\alpha,c}(\kappa,\lambda) \in cD(0).$$

Similarly, for all $\lambda \in \mathbb{Z}[i]$, the proposition $\kappa = [r_c(\lambda)]$ is equivalent to the proposition

$$s_{\alpha,c}(\kappa,\lambda) \in \alpha D(0).$$

Hence, we focus now on possible values of $s_{\alpha,c}(\kappa,\lambda)$ that belong to $cD(0)$ or $\alpha D(0)$.

### 4.2. (Reduced) sets of remainders

Since $gcd(\alpha, c) = \gamma$ due to Eq. 4, $\gamma$ divides $\alpha$ and $c$. Furthermore, for all $\kappa, \lambda \in \mathbb{Z}[i]$, $\gamma$ also divides $s_{\alpha,c}(\kappa, \lambda)$. Thus, we have :

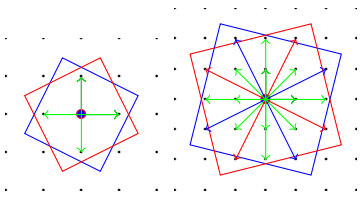
$$\forall (\kappa, \lambda) \in \mathbb{Z}[i] \times \mathbb{Z}[i], s_{\alpha,c}(\kappa, \lambda) = \gamma s_{\gamma,\bar{\gamma}}(\kappa, \lambda). \quad (7)$$

Since $gcd(\gamma, \bar{\gamma}) = 1$ and from Bézout's identity, there exists a family of solutions $\{(\kappa_0 + \tau\bar{\gamma}, \lambda_0 + \tau\gamma)\}$, $\tau \in \mathbb{Z}[i]$, to the equation

$$s_{\gamma,\bar{\gamma}}(\kappa, \lambda) = \kappa \cdot \gamma - \lambda \cdot \bar{\gamma} = 1. \quad (8)$$

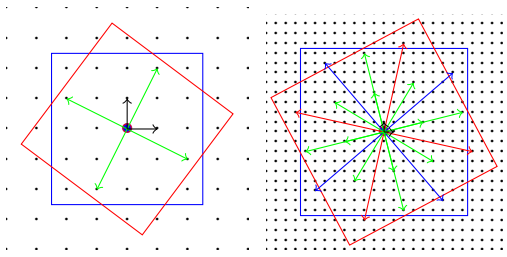
We can conclude that for all $\kappa, \lambda \in \mathbb{Z}[i]$, $s_{\gamma,\bar{\gamma}}(\kappa, \lambda)$ can have any possible values, whereas multiples of $\gamma$ are the only possible values of $s_{\alpha,c}(\kappa, \lambda)$.

Let $S_{\bar{\gamma}}$ (resp. $S_\gamma$) be the reduced set of remainders defined such that $S_{\bar{\gamma}} = \{\rho \in \mathbb{Z}[i] \mid \rho \in \bar{\gamma}D(0)\}$ (resp. $S_\gamma = \{\rho \in \mathbb{Z}[i] \mid \rho \in \gamma D(0)\}$). As illustrated in Fig. 4, these two sets are two sets of integer points lying into two different squares. As illustrated in Fig. 5, there is no loss of generality to compare these reduced sets.



(a) (b)

**Figure 4:** *In (a), the reduced sets of remainders $S_{2+i}$ and $S_{2-i}$. In (b), the reduced sets of remainders $S_{4+i}$ and $S_{4-i}$. Note that $S_{2+i} = S_{2-i}$ but $S_{4+i} \neq S_{4-i}$.*



(a) (b)

**Figure 5:** *In (a), the sets of multiples of $(2+i)$ that belong to $(3+4i)D(0)$ and $5D(0)$. In (b), the sets of multiples of $(4+i)$ that belong to $(15+8i)D(0)$ and $17D(0)$.*

It remains to compare the two reduced sets of remainders $S_{\bar{\gamma}}$ and $S_\gamma$, because the discretized rotation $[r_\alpha]$ is bijective iff $S_{\bar{\gamma}} = S_\gamma$.

### 4.3. Geometry of the reduced sets of remainders

We first show that $S_{\bar{\gamma}} \neq S_\gamma$ if $p > q+1$ (i). Then, we show that $S_{\bar{\gamma}} = S_\gamma$ if $p = q+1$ (ii).

To show (i), we exhibit a Gaussian integer that belongs to $S_\gamma$ but not to $S_{\bar{\gamma}}$ if $p > q+1$. Without loss of generality, we multiply everything by $(1+i)$ so that the vertices of the discretization cells $(1+i)\gamma D(0)$ and $(1+i)\bar{\gamma}D(0)$ are Gaussian integers (see Fig. 6). Let $\zeta$ be equal to $\gamma - 1 = (p-1) + qi$. It is easy to see that $\zeta \in (1+i)\gamma D(0)$. We now want to show that it does not belong to $(1+i)\bar{\gamma}D(0)$ because

$$(p+q)(p-1) + (p-q)(q-1) - p^2 - q^2 > 0, \quad (9)$$

i.e. $i\bar{\gamma}$, $\bar{\gamma}$ and $\zeta$ are counter-clockwise oriented.

Developing Eq. 9 we get

$$2q(p-q) - p - q > 0.$$

If we write $p = q + e$, we obtain :

$$e > \frac{2q}{2q-1},$$

which is always true if $q > 1$ and $e > 1$ or if $q = 1$ and $e > 2$.

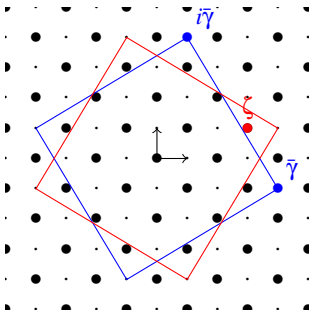



**Figure 6:** *Example for $\gamma = 4+i$. The discretization cells $(1+i)\gamma D(0)$ and $(1+i)\bar{\gamma}D(0)$ are respectively depicted with red and blue. The images of $i\bar{\gamma}$ and $\bar{\gamma}$ are depicted with blue dots, whereas $\zeta = \gamma - 1$ is depicted with a red dot. These three points are counter-clockwise oriented, which means that $S_\gamma \neq S_{\bar{\gamma}}$ for the pair $p = 4$ and $q = 1$.*

To show (ii), it is enough to see that if $p = q+1$, the boundaries of the discretization cells $\gamma D(0)$ and $\bar{\gamma}D(0)$ lie between two consecutive $L_1$ balls of integral radius $q$ and $p = q+1$, which means that $S_{\bar{\gamma}} = S_\gamma$ (see Fig. 4).

From (i) and (ii), we finally have theorem 1.

### Références

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